Math 852 Homework 16

You may check your calculations with a computer algebra system, but they must be done by hand to receive any credit!

1. Let $M$ be an $R$-module and let $M_1, M_2$ be submodules so that $M = M_1 \oplus M_2$. Let $\pi_i \in \text{Hom}_R(M, M_i)$ be the projection maps.

(a) Let $\varphi \in \text{Hom}_R(M, M)$ and prove that $\varphi = \sum_{i,j=1}^{2} \pi_i \circ \varphi \circ \pi_j$. Set $\varphi_{i,j} = \pi_i \circ \varphi \circ \pi_j|_{M_j}$. One can define a map

$$
\varphi \mapsto \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix}
$$

where $\varphi_{i,j} \in \text{Hom}_R(M_j, M_i)$. Show that the usual operations of matrix addition and multiplication make the set of all such matrices into a ring and that the above map is then a ring isomorphism.

(b) Interpret this in the case that $M$ is a 2-dimensional vector space over a field. Generalize to the case of $n$ summands.

(c) Determine $\text{Hom}_Z(A, A)$ when $A = \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

2. Let $V$ be a 3-dimensional vector space over $\mathbb{F}_2$ considered as a $\mathbb{F}_2[x]$-module via $T \in \text{Hom}_{\mathbb{F}_2}(V, V)$ defined on basis elements $v_1, v_2, v_3$ as follows:

$$
T(v_1) = v_1 + v_3 \\
T(v_2) = v_1 + v_2 \\
T(v_3) = v_2 + v_3.
$$

Determine $\text{Ann}_{\mathbb{F}_2}(v)$ for each $v \in V$. Use this to deduce that $V$ is a torsion module. Find a nonzero $f \in \mathbb{F}_2[x]$ so that $fV = 0$.

3. Calculate the invariant factor matrix over $\mathbb{Q}[x]$ for

$$
A = \begin{pmatrix} 1 - x & 1 + x & x \\ x & 1 - x & 1 \\ 1 + x & 2x & 1 \end{pmatrix}.
$$
4. Let $A$ be given by
\[
\begin{pmatrix}
2 & 1+i & 1-i \\
8+6i & -4 & 0
\end{pmatrix}.
\]
Find square matrices $S$ and $T$ over $\mathbb{Z}[i]$ so that $SAT$ is an invariant factor matrix for $A$. What is the answer if one works over $\mathbb{C}$ instead of $\mathbb{Z}[i]$?

5. Let
\[
A = \langle a, b, c : -4a + 2b + 6c = -6a + 2b + 6c = -7a + 4b + 15c = 0 \rangle.
\]
Show that $|A| = 12$. Find elements $u$ and $v$ in $A$, of orders 2 and 6 respectively, such that $A = \langle u \rangle \oplus \langle v \rangle$.

6. Suppose that $R = (r_{ij}) \in \text{GL}_n(\mathbb{Z})$. What can you say about the group
\[
\left\langle a_1, \ldots, a_n : \sum_{j=1}^{n} r_{ij}a_i = 0 \text{ for } j = 1, \ldots, n \right\rangle?
\]

7. Suppose that an abelian group $G$ is generated by $x, y$ and $z$ satisfying the following relations:
\[
\begin{align*}
9x - 12y + 12z &= 0 \\
12x - 18y + 18z &= 0.
\end{align*}
\]
What is the group structure of $G$ (as a sum of cyclic groups)?

8. Determine all group homomorphisms from $\mathbb{Z}/6\mathbb{Z}$ to $\mathbb{Z}/15\mathbb{Z}$. 