
2. Let $G$ be the group $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let $N$ be the cyclic subgroup $\langle (1,1) \rangle$. Describe the quotient group $G/N$.

3. Let $G$ be a group and let $A$ be the subset of $G$ consisting of elements of the form $xyx^{-1}y^{-1}$. Let $[G,G]$ denote the subgroup of $G$ generated by $A$. This subgroup is referred to as the commutator subgroup of $G$.

   (a) Prove that $[G,G]$ is normal in $G$.

   (b) Prove that $G/[G,G]$ is abelian.

4. Define a map $\varphi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ by $n \mapsto (n + 3\mathbb{Z}, n + 6\mathbb{Z})$. Prove that $\varphi$ is a homomorphism. Is it surjective? What is the kernel of $\varphi$?

5. Let $p$ be a prime. Prove that $a^{p-1} \equiv 1 \pmod{p}$ for all $a \in \mathbb{Z}$ with $\gcd(a,p) = 1$.

6. Let $G$ be a group. Let $K$ be a subgroup of $G$ and let $K \setminus G$ denote the set of right cosets.

   (a) If $g \in G$, show that the map $\phi_g : K \setminus G \to K \setminus G$ given by $\phi_g(Kb) = Kb_g$ is a permutation of the set $K \setminus G$.

   (b) Prove that the function $\psi : G \to \text{Sym}(K \setminus G)$ given by $\psi(g) = \phi_{g^{-1}}$ is a homomorphism of groups with kernel contained in $K$.

   (c) If $K$ is normal in $G$, prove that $K = \ker(\psi)$.

   (d) Use these results to prove Cayley’s theorem, namely, that every group is isomorphic to a group of permutations.

7. Let $N \in \mathbb{Z}_{>1}$ and define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$
(a) Prove that $\Gamma_0(N)$ is a subgroup of $\text{SL}_2(\mathbb{Z})$.

(b) Prove that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\}.$$ 

(c) Describe the quotient group $\Gamma_0(N)/\Gamma_1(N)$.

8. Let $G$ be a group.

(a) Prove that if $G/Z(G)$ is cyclic then $G$ is abelian.

(b) Suppose that $|G| = pq$ for $p$ and $q$ primes. Prove that either $G$ is abelian or $Z(G) = 1$. 

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