Math 851 Homework 5

1. (a) Let $p$ be a prime and $U_n(\mathbb{Z}/p\mathbb{Z})$ the subset of $\text{GL}_n(\mathbb{Z}/p\mathbb{Z})$ given by

$$U_n(\mathbb{Z}/p\mathbb{Z}) = \{(x_{ij}) \in \text{GL}_n(\mathbb{Z}/p\mathbb{Z}) : x_{ij} = 0 \text{ for all } i > j, \ x_{ii} = 1 \text{ for all } i\}.$$

Show that $U_n(\mathbb{Z}/p\mathbb{Z})$ is a $p$-Sylow subgroup of $\text{GL}_n(\mathbb{Z}/p\mathbb{Z})$. It may be helpful to know that

$$|\text{GL}_n(\mathbb{Z}/p\mathbb{Z})| = \prod_{i=0}^{n-1} (p^n - p^i).$$

(b) Prove that the number of $p$-Sylow subgroups of $\text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is $p + 1$.

2. Prove that if $n_p(G) \not\equiv 1 \pmod{p^2}$, then there are distinct $p$-Sylow subgroups $P$ and $Q$ of $G$ so that

$$[P : P \cap Q] = [Q : P \cap Q] = p.$$

3. Prove that if $|G| = 1365$ then $G$ is not simple.

4. Let $P \in \text{Syl}_p(G)$ and assume $N$ is normal in $G$. Prove that $P \cap N$ is a $p$-Sylow subgroup of $N$. Deduce that $PN/N$ is a $p$-Sylow subgroup of $G/N$.

5. If $p$, $q$, and $r$ are distinct primes, prove that a group of order $pqr$ is not simple.

6. Let $G$ be a group and suppose that every Sylow subgroup of $G$ is normal. Prove that $G$ is a direct product of its Sylow subgroups, one for each prime that divides $|G|$.

7. Let $P \in \text{Syl}_p(G)$ and $N$ a normal subgroup of $G$. If $P$ is normal in $N$, prove that $P$ is normal in $G$.

8. Let $n \in \mathbb{Z}_{\geq 1}$. Prove that $S_n$ is isomorphic to a subgroup of $\text{GL}_n(\mathbb{Z})$. Use this to show that every finite group is isomorphic to a subgroup of $\text{GL}_n(\mathbb{Z})$ for some positive integer $n$. 