Math 851 Homework 9

All rings \( R \) are assumed to be commutative with \( 1_R \neq 0 \).

1. (a) Prove that \( \mathbb{Z}[\sqrt{-2}] \) is a Euclidean domain. (Adapt the proof that we used for \( \mathbb{Z}[i] \). Plot points of \( \mathbb{Z}[\sqrt{-2}] \) in \( \mathbb{C} \) to get a better idea of what is going on.)

(b) Use the Euclidean algorithm to find the greatest common divisor of \( 3 - 7\sqrt{-2} \) and \( 4 + 3\sqrt{-2} \) in \( \mathbb{Z}[\sqrt{-2}] \).

2. Recall that a ring \( R \) is said to satisfy the ascending chain condition on ideals if for every chain of ideals

\[
I_1 \subset I_2 \subset I_3 \subset \cdots
\]

there is a \( n \in \mathbb{Z}_{\geq 1} \) so that \( I_n = I_{n+j} \) for all \( j \geq 0 \). Prove that a ring \( R \) satisfies the ascending chain condition on ideals if and only if every ideal in \( R \) is finitely generated.

3. Let \( R = \mathbb{Z}[\sqrt{-5}] \). Set \( I_1 = (2, 1 + \sqrt{-5}), I_2 = (3, 2 + \sqrt{-5}), \) and \( I_3 = (3, 2 - \sqrt{-5}) \).

(a) Prove that 2, 3, and \( 1 \pm \sqrt{-5} \) are irreducible elements in \( R \), no two of which are associate. Thus,

\[6 = 2 \cdot 3 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})\]

are two distinct factorizations of 6 and so \( R \) is not a UFD.

(a) Prove that \( I_1, I_2, \) and \( I_3 \) are not principal ideals in \( R \).

(b) Prove that the product of two nonprincipal ideals can be principal by showing \( I_1^2 = (2) \).

(c) Prove that \( I_1, I_2, \) and \( I_3 \) are all prime ideals in \( R \). (Hint: For example, for \( I_2 \), observe we have \( R/I_2 \cong (R/(3))/(I_2/(3)) \) by the third isomorphism theorem. Show that \( R/(3) \) has 9 elements, \( I_2/(3) \) has 3 elements, and so \( R/I_2 \) has three elements.)
(d) Show that the factorizations in (a) imply that $(6) = (2)(3)$ and $(6) = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Show that these two ideal factorizations give the same factorization of the ideal $(6)$ as the product of prime ideals by showing $(6) = I_1^2I_2I_3$ and relating this to the two ideal factorizations.

4. Determine all the ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.

5. Note that for this problem it may help to read the notes on $\mathbb{Z}[i]$ that were not covered in class.

(a) Prove that the quotient ring $\mathbb{Z}[i]/(1 + i)$ is a field of order 2.

(b) Let $p \in \mathbb{Z}$ be a prime with $p \equiv 3 \pmod{4}$. Prove that the ring $\mathbb{Z}[i]/(p)$ is a field with $p^2$ elements.

(c) Let $p \in \mathbb{Z}$ be a prime with $p \equiv 1 \pmod{4}$. Write $p = \varpi \overline{\varpi}$ for $\varpi = a + bi$ a prime in $\mathbb{Z}[i]$ and $\overline{\varpi} = a - bi$. Show that the hypotheses for the Chinese Remainder Theorem are satisfied and that

$$\mathbb{Z}[i]/(p) \cong \mathbb{Z}[i]/(\varpi) \times \mathbb{Z}[i]/(\overline{\varpi})$$

as rings. Show that the quotient ring $\mathbb{Z}[i]/(p)$ has $p^2$ elements and conclude that $\mathbb{Z}[i]/(\varpi)$ and $\mathbb{Z}[i]/(\overline{\varpi})$ are each fields of order $p$.

6. Let $R$ be a PID. Is the quotient of $R$ by an ideal necessarily a PID? What if the ideal is prime? If the answer is no, give a counterexample. If it is yes, prove it.

7. Show that the radical of $(x, y^2)$ in $\mathbb{Q}[x, y]$ is $(x, y)$. Deduce that $(x, y^2)$ is a primary ideal that is not a power of a prime ideal.