MATH 852 — MIDTERM EXAM
March 1, 2011

NAME: ________________________________

1. Do not open this exam until you are told to begin.
2. This exam has 13 pages including this cover. There are 4 problems.
3. Write your name on the top of EVERY sheet of the exam!
4. If you separate pages of this exam and include additional pages, please be sure to staple them in the correct order before turning the exam in!
5. You may quote major theorems, but nothing that trivializes a problem. If you are unsure if you are allowed to quote a theorem, ask.

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1. (5+8+8+8 points) Let $F = \mathbb{Q}(\sqrt[4]{2})$.

(a) Is $F/\mathbb{Q}$ Galois? Justify your answer.

(b) Let $K$ be the Galois closure of $F/\mathbb{Q}$. What is $\deg(K/\mathbb{Q})$? What polynomial is $K$ the splitting field of?
(c) Let $\mathbb{F}$ be a finite field of characteristic $p$. Let $g(x)$ be an irreducible separable polynomial over $\mathbb{F}$ of degree $n$. Show that the Galois group of $g$ over $\mathbb{F}$ is cyclic of order $n$. (You may use any facts about finite fields and their Galois groups that we proved in class.)
(d) Determine the splitting field and Galois group of \( x^4 - 2 \) considered over \( \mathbb{F}_3 \). (Hint: Is \( f(x) \) irreducible?)
2. (10 points) Let $f(x) \in \mathbb{Q}[x]$ and let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Suppose that $\deg(K/\mathbb{Q}) = 1225$. Show that $f$ is solvable by radicals. You may use basic results from Math 851 that do not trivialize the problem, but be sure to carefully state any results you use.
3. (8+10+8+6 points each) Let $f(x) = x^3 - 7$.

(a) Find the splitting field of $f$ over $\mathbb{Q}$. Denote this field as $K$ for the remainder of the problem.
(b) Find $\text{Gal}(K/Q)$. Work out the details here please. In particular, it is not enough to give an abstract group isomorphism. You must actually give the automorphisms in $\text{Gal}(K/Q)$. 

(c) Draw the subgroup and corresponding subfield diagrams for \( \text{Gal}(K/\mathbb{Q}) \).
(d) Prove that the polynomial $g(x) = x^2 - 2x - 1$ is irreducible over $K$. 
4. (8+7+6+8 points) Let $f(x) = 2x^5 - 10x + 5$. Let $K$ be the splitting field of $f$ over $\mathbb{Q}$.
   (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$. Conclude that $\text{Gal}(K/\mathbb{Q})$ has an element $\sigma$ of order 5.
(b) Prove that \( f \) has exactly 3 distinct real roots. You can use calculus to show this. (Hint: How would you graph this in calculus class? Use that information.)
(c) Given a polynomial $g(x) \in \mathbb{Q}[x]$ and $\alpha \in \mathbb{C} - \mathbb{R}$ a root of $g(x)$, prove that $\overline{\alpha}$ is a root of $g(x)$. Use this to show that complex conjugation gives a nontrivial element $c \in \text{Gal}(K/\mathbb{Q})$ of order 2.
(d) We have an element of order 5 and an element of order 2 in $\text{Gal}(K/Q)$. Since $\text{Gal}(K/Q)$ permutes the roots of a degree 5 polynomial, we can realize $\text{Gal}(K/Q)$ as a subgroup of $S_5$. We show $f(x)$ is not solvable by radicals by showing $\text{Gal}(K/Q) \cong S_5$. Without loss of generality and identifying $\text{Gal}(K/Q)$ with its image in $S_5$, we can assume $c = (12)$ and $\sigma = (12345)$ are elements in $\text{Gal}(K/Q)$. First, by considering $\sigma^k c \sigma^{-k}$ for $k \geq 1$, show that $(23)$, $(34)$, and $(45)$ are in $\text{Gal}(K/Q)$. Use this to show that the rest of the transpositions are in $\text{Gal}(K/Q)$. This gives that $\text{Gal}(K/Q)$ is $S_5$ upon recalling every element of $S_n$ can be written as a product of transpositions.