MATH 853 — Final Exam
December 14, 2012

NAME: ___________________________________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 4 problems.
3. Write your name on the top of EVERY sheet of the exam at the START of the exam!
4. Do not separate the pages of the exam.
5. If you are unsure if you are allowed to use a theorem, ask.
6. Turn off all cell phones.

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1. (4+4+4+5+8 points) (a) Let $V$ and $W$ be finite dimensional vector spaces over a field $F$ and let $T \in \text{Hom}_F(V,W)$. State the rank-nullity theorem for $T$, i.e., give a relation between the dimensions of $\ker(T)$ and $\text{Im}(T)$ in terms of $\dim_F(V)$ or $\dim_F(W)$.

(b) For $A \in \text{Mat}_n(F)$, let $\text{tr}(A)$ denote the trace of $A$. (You can use the definition from our homework, or the undergraduate definition.)

(i) Prove that $\text{tr} \in \text{Mat}_n(F)^*$. (You are allowed to use familiar properties of the trace map.)
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(ii) Let $\mathfrak{sl}_n(F)$ denote the space of trace zero matrices in $\text{Mat}_n(F)$. Determine the dimension of $\mathfrak{sl}_n(F)$.

(iii) Find a basis for $\mathfrak{sl}_n(F)$. 

(c) Let $\phi \in \text{Mat}_n(F)^*$ satisfy
1. $\phi(AB) = \phi(BA)$ for all $A, B \in \text{Mat}_n(F)$.
2. $\phi(I_n) = n$ where $I_n \in \text{Mat}_n(F)$ is the identity matrix.
Prove that $\phi = \text{tr}$. 
2. (10 +15 points) (a) Let $V = \mathbb{F}_3^3$ and let $\mathcal{E}_3$ be the standard basis for $V$. Let $\varphi : V \rightarrow V$ be the symmetric bilinear form given by

$$[\varphi]_{\mathcal{E}_3} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$ 

Find an orthogonal basis for $V$ with respect to $\varphi$. Can you make this an orthonormal basis? Be sure to justify your answer.
(b) Classify all nondegenerate symmetric bilinear forms on $V$. In other words, give a finite collection of matrices so that every nondegenerate symmetric bilinear form on $V$ is isometric to exactly one of the forms listed. Give a short justification on why your list is complete.
3. (15+10 points) (a) Find the rational canonical form of the matrix

\[
A = \begin{pmatrix}
1 & 2 & -4 & 4 \\
2 & -1 & 4 & -8 \\
1 & 0 & 1 & -2 \\
0 & 1 & -2 & 3
\end{pmatrix}.
\]
(b) Prove there are no matrices $A \in \text{Mat}_3(\mathbb{Q})$ so that $A^8 = I_3$ but $A^4 \neq I_3$. What about if one works over $\mathbb{C}$ instead of $\mathbb{Q}$?
4. (4+5+8+8 points) Let \((V, \varphi)\) be a complex inner product space with \(\dim_{\mathbb{C}} V = n\).

(a) Define what it means for \(T \in \text{Hom}_{\mathbb{C}}(V, V)\) to be a normal map.

(b) Translate the above definition into matrix language. You do not need to prove your answer, but a short explanation is required.
(c) Prove that if $T \in \text{Hom}_\mathbb{C}(V, V)$ is normal, then there exists a map $S \in \text{Hom}_\mathbb{C}(V, V)$ so that $T = S \circ S$. 
(d) Let $A \in \text{Mat}_n(\mathbb{C})$ have $n$ linearly independent eigenvectors. Prove that $A$ is normal iff each eigenvector of $A$ is an eigenvector of $A^\dagger := \overline{\langle A \rangle}$. 