MATH 853 — Midterm
October 23, 2012

NAME: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 4 problems.
3. Write your name on the top of EVERY sheet of the exam at the start of the exam!
4. Do not separate the pages of the exam.
5. If you are unsure if you are allowed to use a theorem, ask.
6. Turn off all cell phones.

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1. (5 + 5 + 15 points) Let $F = \mathbb{F}_7$. Recall we defined for a positive integer $n$ a vector space $P_n(F)$ by

\[ P_n(F) = \{ f(x) \in F[x] : \deg(f) \leq n \}. \]

(a) What is the dimension of $P_n(F)$ over $F$? How many elements are in $P_n(F)$? Give a short justification for your answer.

(b) Let $V = P_3(F)$. Consider bases of $V$ given by $B = \{1, x, x^2, x^3\}$ and $C = \{1, (x+1), (x+1)^2, (x+1)^3\}$.
Define $T \in \text{Hom}_F(V, V)$ by

\[
T(1) = 1 + (x + 1) + 3(x + 1)^2 + 2(x + 1)^3, \\
T(x) = 4 + (x + 1) + 5(x + 1)^2 + 2(x + 1)^3, \\
T(x^2) = 3(x + 1) + 6(x + 1)^2 + 6(x + 1)^3, \\
T(x^3) = 1 + (x + 1) + 2(x + 1)^2 + 2(x + 1)^3.
\]

Determine the matrix $[T]_B^C$. 

(c) Give a basis for the kernel and image of $T$. 
2. (10+15 points) Let $V$ be a finite dimensional $F$-vector space. Let $T \in \text{Hom}_F(V,V)$.

(a) Define the minimal polynomial $m_T(x)$ of $T$. Given $v \in V$, define the annihilating polynomial $m_{T,v}(x)$. Prove that for any $v \in V$, $m_{T,v}(x) \mid m_T(x)$. 

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Recall that we say $\lambda \in F$ is an eigenvalue of $T$ if there exists a nonzero vector $v \in V$ so that $T(v) = \lambda v$. We call such a $v$ an eigenvector.

(b) Prove that $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is a root of the minimal polynomial $m_T(x)$. 


3. (25 points) Let $F$ be a field. Prove that

$$F[x] \otimes_F F[y] \cong F[x, y].$$
4. (8 + 8 + 9 points) (a) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear map so that

$$
[T]_{\mathcal{E}_3} = \begin{pmatrix}
0 & 2 & 0 \\
1 & 1 & 1 \\
0 & 3 & 2
\end{pmatrix}
$$

where $\mathcal{E}_3 = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Compute the determinant of $T$ by using the definition of the determinant given in this class.
(b) Give a basis for $\Lambda^2(\mathbb{R}^3)$. Calculate the matrix of $\Lambda^2(T)$ with respect to this basis.
(c) Now consider finite dimensional $F$-vector spaces $V$ and $W$. Let $T \in \text{Hom}_F(V,W)$. Prove that if $T$ is a surjection, then $\Lambda^k(T)$ is a surjection for all $k \geq 1$. Prove that if $T$ is an isomorphism, then $\Lambda^k(T)$ is an isomorphism for all $k \geq 1$. 