Chapter D Review of Complex Analysis:

The subject of complex analysis is a vast and beautiful subject that should certainly be studied for its own right. However, we need to use complex analysis heavily in this course and as it is not a required course at Clemson, we present this very terse overview/review to attempt to give a bare minimum needed to proceed in the course.

The complex numbers are certainly familiar. One can define them formally as

\[ \mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \} \]

and endow them with addition and multiplication via

\[ (a + bi) + (c + di) = (a + c) + (b + d)i \]
\[ (a + bi)(c + di) = (ac - bd) + (ad + bc)i. \]

One then checks that \( \mathbb{C} \) forms a field containing \( \mathbb{R} \).

One also can define \( \mathbb{C} \) to be the degree 2 field extension of \( \mathbb{R} \) given by \( \mathbb{R}(x)/\langle x^2 + 1 \rangle \).
As an \( \mathbb{R}^2 \)-vector space, we have \( \mathbb{R}^2 \cong \mathbb{C} \). This allows us to define for \( z = x + iy \in \mathbb{C} \),

\[ |z| = (x^2 + y^2)^{1/2}. \]

The complex conjugate of \( z \) is \( \bar{z} = x - iy \). Note

\[ |z|^2 = z \bar{z}. \]

We can represent \( \mathbb{C} \) as \( \mathbb{R}^2 \) for geometric purposes as well.

Consider \( w = \frac{z}{|z|} \). Then \( |w| = 1 \), so \( w \) lies on the unit circle.

This allows us to write

\[ w = \cos \theta + i \sin \theta. \]

for some \( \theta \in \mathbb{R} \). Define \( e^{i\theta} = \cos \theta + i \sin \theta \). Then we have

\[ z = |z| e^{i\theta}. \]

We typically write \( |z| = r \), so \( z = re^{i\theta} \).
We define \( \arg z \) to be the value of \( \theta \) so that
\[ z = re^{i\theta}. \] Note this is not a well-defined function as if
\[ z = re^{i\theta}, \] then \( z = re^{i(\theta + 2\pi n)} \) for any \( n \in \mathbb{Z} \). If we wish to
restrict to a particular range of values for \( \theta \), we use a branch of \( \arg \). For instance, if we want \( 0 \leq \theta < 2\pi \), we
use \( \text{Arg}_0 z \). This is the value of \( \arg z \) that lies between
0 and \( 2\pi \). One should be aware this causes many subtleties.

For instance, the logarithm function is defined by
\[
\log z = \ln |z| + i \arg z, \quad z \neq 0
\] as is not a well-defined function either. If we want to work
with a function, we must pick a branch here as well. For
instance,
\[
\text{Log}_0 z = \ln |z| + i \text{Arg}_0 z.
\]

Recall from calculus that \( x^a = e^{a \log x} \). We define
powers here via
\[
z^a = e^{a \log z}
\] for \( a \in \mathbb{C} \) where \( e^{x+iy} = e^x e^{iy} \). Again we see the
is not a function so we must choose a branch here as well. These are just some things to keep in mind.

Complex analysis is the study of calculus for functions $f : \mathbb{C} \to \mathbb{C}$. Continuity is defined topologically, so that does not change. We define differentiability the same as well. However, given $f : \mathbb{C} \to \mathbb{C}$, we can write this as

$$f(z) = u(x,y) + i v(x,y)$$

for functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. This allows us to reduce questions about differentiability of $f$ to questions about differentiability of $u$ and $v$. In particular, we have the following result:

**Theorem 0.1:** Let the function $f(z) = u(x,y) + i v(x,y)$ be defined in some $\mathcal{E}$- nbhd of a point $z_0 = x_0 + iy_0$, and suppose the $1$st order partial derivatives of the fields $u$ and $v$ with respect to $x$ and $y$ exist everywhere in that nbhd. If these partial derivatives are continuous at $(x_0, y_0)$ and satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$
at \((x_0, y_0)\), then \(f'(z_0)\) exists and is given by

\[
f'(z_0) = U_x(x_0, y_0) + i V_x(x_0, y_0).
\]

In general, we aren't interested in functions that are just differentiable at a point \(z_0\); we want them to be analytic at \(z_0\).

**Def:** We say a function \(f(z)\) is analytic at \(z_0\) if there exists \(\varepsilon > 0\) so that \(f'(z)\) exists at every \(z\) with \(|z - z_0| < \varepsilon\).

We say \(f\) is analytic on an open nbhd if \(f\) is analytic at every point in the nbhd. We say \(f\) is entire if it is analytic on \(\mathbb{C}\). We say \(z_0\) is a singular point if \(f\) is not analytic at \(z_0\) but is analytic at some point in every nbhd of \(z_0\).

Analytic functions are extremely nice to work with. For example, if \(D\) is open and connected and \(f\) is analytic on \(D\), then \(f\) is uniquely determined by its values on any line segment in \(D\) or any open ball in \(D\).

In terms of singular points, we will be most interested in poles. We say \(z_0\) is a pole of order \(m\) of \(f(z)\) if there
exists a function \( f(z) \) that is analytic at and meromorphic at \( z = 0 \) so that

\[
f(z_1) = \frac{f(z)}{(z-z_0)^m}.
\]

We will also make heavy use of integrals. Let \( f(z) \) be a complex function. Let \( C: z(t) \) be a contour given by \( a \leq t \leq b \). We define the integral of \( f \) along \( C \) by

\[
\int_C f(z) \, dz = \int_a^b f(z(t)) \, z'(t) \, dt.
\]

This reduces integrating a complex function along a contour to calculating a real integral. However, the real power of complex analysis comes when \( C \) happens to be a simple positively oriented closed curve. In this case one has very powerful theorems. We now list some of them. Proofs can be found in any introductory complex analysis book.

**Theorem 0.2:** Let \( f \) be continuous on a domain \( D \). If any one of the following statement is true, so are the others:

1) \( f(z) \) has an antiderivative \( F(z) \) on \( D \);
2) the integrals of $f(z)$ along contours lying entirely in $D$ and extending from any fixed point $z_1$ to any fixed point $z_2$ all have the same value;

3) the integrals of $f(z)$ around closed contours lying entirely in $D$ all have value 0.

**Theorem 0.3:** Suppose that

1) $C$ is a simple closed contour, oriented positively.

2) $C_k$, $k = 1, \ldots, n$ are simple closed contours interior to $C$, all positively oriented, that are disjoint and whose interiors have no points in common.

If $f$ is analytic on all these contours and throughout the multiply connected domain consisting of points between $C$ and the $C_k$, then

$$
\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz.
$$

This theorem is very useful. It allows one to deform complicated contours to simple ones such as circles.

**Theorem 0.4 (Cauchy integral formula):** Let $f(z)$ be analytic everywhere inside and on a simple closed contour $C$ positively oriented.

If $z$ is any point interior to $C$ then
\[ f(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w) \, dw}{(w-z)^{n+1}}. \]

This has many uses. It is used to prove:

1) The fundamental theorem of algebra

2) Maximum modulus principle: if \( f \) is analytic and nonconstant on a domain \( D \), then \( |f(z)| \) has no maximum value on \( D \).

Just as in calculus, if \( f \) is analytic at a point \( z_0 \) then one has a Taylor series that converges for all \( z \) with \( |z-z_0| < R \) for some \( R > 0 \):

\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n. \]

We do get a little more here though. Suppose \( f \) is analytic in an annular domain \( R_1 < |z-z_0| < R_2 \), centered at \( z_0 \), and let \( C \) be any positively oriented simple closed contour around \( z_0 \) lying in the domain. Then for each \( z \) in the domain, \( f(z) \) has a series representation:

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=-\infty}^{0} a_n (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n. \]

where

\[ a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) \, dz}{(z-z_0)^{n+1}}. \]
This is referred to as the Laurent series of \( f \). One easily shows this recession the Taylor series of \( f \) is analytic on \( |z - z_0| < R \).

The value of \( a_n \) is particularly important. Note that

\[
2\pi i \, a_n = \int_{C} f(z) \, dz.
\]

We refer to \( a_n \) as the residue of \( f \) at \( z_0 \) and write \( \text{Res}(f, z_0) \) for \( a_n \). Residues have many applications. For instance, one can use them to calculate many real integrals. We won't need that for the course however.