Chapter 2  Functions and Maps on Riemann Surfaces:

The main point of this chapter is to show how to transfer some of the basic definitions and results of complex analysis over to the theory of Riemann surfaces. Before we do this we introduce the notion of a sheaf. One generally sees these in algebraic geometry, but they also arise in the study of complex function theory. Since we may run into sheaves in the future we'll encounter from sheaves, we might as well introduce them now.

**Definition:** Let $X$ be a topological space. A *presheaf of groups* (modules, algebras, etc.) $\mathcal{F}$ on $X$ is a collection of groups $\mathcal{F}(U)$, one for each open set $U \subseteq X$ and a collection of group homomorphisms $\rho^U : \mathcal{F}(U) \to \mathcal{F}(V)$, where $V \subseteq U$ so that

- $\mathcal{F}(\emptyset)$ is the trivial group (modules, algebras, etc.)
- $\rho^U = \text{id}$
- if $W \subseteq V \subseteq U$, then $\rho^W = \rho^V \circ \rho^U$.

The maps $\rho^U$ are called *restriction maps*. The elements of
$\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over $U$ and the elements of $\mathcal{F}(x)$ are called the global sections.

Examples: 1. Let $X$ be any topological space and set

$$\mathcal{O}_x(U) = \{\text{continuous functions } f: U \to \mathbb{C}\}.$$  

Then $\mathcal{O}_x$ forms a presheaf of algebras on $X$ with the "restriction maps" being given by restriction of functions.

2. Let $X = \mathbb{C}$ and set

$$\mathcal{O}_x(U) = \{\text{holomorphic functions } f: U \to \mathbb{C}\}.$$  

Then $\mathcal{O}_x$ is a sheaf of rings on $X$.

While presheaves are nice, what one sees is that one has a presheaf anytime one has a property that is preserved under restriction to an open set. We will be more interested in properties that can be checked locally. Namely, we want properties that satisfy if $U = U_1 \cup U_i$, then the property holds on $U$ iff it holds on each $U_i$. This brings us to the definition of a sheaf.
Def: Let \( \mathcal{F} \) be a pretopos on \( \mathbb{X} \), \( U \subseteq \mathbb{X} \) an open set, and \( \mathcal{U} = \{U_i\} \) an open covering of \( U \). We say \( \mathcal{F} \) is a \underline{sheaf} if for each such \( U \) and \( \mathcal{U} \), if whenever one has \( s_i \in \mathcal{F}(U_i) \) so that the \( s_i \) agree on the intersections, i.e.,

\[
\mathcal{F}_i(U_i \cap U_j)(s_i) = \mathcal{F}_j(U_i \cap U_j)(s_j) \quad \forall i, j,
\]

Then these sections patch together uniquely to give a section \( s \in \mathcal{F}(U) \) so that \( \mathcal{F}_i(U)(s) = s_i \quad \forall i \).

There are other useful ways of phrasing this. For instance, if \( \mathcal{F} \) is a sheaf and \( s, t \in \mathcal{F}(U) \) so that \( \mathcal{F}_i(s) = \mathcal{F}_i(t) \) for every \( i \), then \( s = t \). One uses this often to conclude a section is exactly \( 0 \).

With this language we may proceed with the material of this chapter. Throughout this chapter, we let \( X \) denote a Riemann surface.

Def: Let \( p \in X \) and \( f \) be a complex-valued function defined in a neighborhood of \( p \). We say \( f \) is \underline{holomorphic at} \( p \).

If there is a chart \( \phi: U \to V \) around \( p \) so that \( \phi(f) \) is holomorphic at \( \phi(p) \). Let \( W \subseteq X \). We say \( f \)
Lemma 2.1: Let \( p \in X \) and \( f : W \rightarrow \mathbb{C} \) when \( W \ni p \). Then

1) \( f \) is holomorphic at \( p \) iff for every chart \( \phi : U \rightarrow V \) with \( p \in U \), the composition \( f \circ \phi^{-1} \) is holomorphic at \( \phi(p) \).

2) \( f \) is holomorphic in \( W \) iff there is a collection of charts \( \{ \phi_i : U_i \rightarrow V_i \} \) so that \( W \subseteq U_i \) and \( f \circ \phi^{-1} \) is holomorphic on \( \phi_i(W \cap U_i) \) for each \( i \).

3) If \( f \) is holomorphic at \( p \), \( f \) is holomorphic in a neighborhood of \( p \).

Proof: Exercise.

Let \( W \subseteq X \) be an open subset. Define

\[ \mathcal{O}_X(W) = \{ f : W \rightarrow \mathbb{C} : f \text{ is holomorphic} \} \]

Then one easily checks \( \mathcal{O}_X \) is a sheaf of \( \mathbb{C} \)-algebras on \( X \).

Examples:
1) Any complex chart considered as a function on its domain is holomorphic.
2) Let $Y$ be an affine plane curve defined by $f(z, w) = 0$. 

Projections to $z$ or $w$ are holomorphic functions. Any polynomial $g(z, w) \in \mathbb{C}[z, w]$ restricted to $X$ is holomorphic.

3) Let $X$ be a projective plane curve given by the homogeneous polynomial $F(x, y, z) = 0$. Let $p = [x_0 : y_0 : z_0] \in X$ with $x_0 \neq 0$. The functions $y/x$ and $z/x$ are holomorphic functions on $Y$ at $p$. Moreover, any polynomial $g(y/x, z/x)$ is holomorphic at $p$ when restricted to $X$. Any such polynomial can be written as $\frac{G(x, y, z)}{x^d}$ for $G(x, y, z)$ the homogenization of $G$ and $\deg G = d$. More generally, if $G$ and $H$ are homogeneous polynomials of the same degree and $H$ does not vanish at $p$ then $G/H$ is holomorphic at $p$ when restricted to $X$.

Now that we have defined holomorphic functions on $X$, the next step is to deal with functions with singularities just as in $\mathbb{C}$. The definitions are exactly what one would expect. Recall a punctured neighborhood around a pt $p$ is given by $U - \{p\}$ where $U$ is an open set.

Def: Let $f$ be holomorphic in a punctured neighborhood of $p \in X$. 


1) We say \( f \) has a \underline{removable singularity} at \( p \) if there exists a chart \( \phi : U \to V \), \( p \in U \) so that \( f \circ \phi^{-1} \) has a \underline{removable singularity} at \( \phi(p) \).

2) We say \( f \) has a \underline{pole} at \( p \) if there is a chart \( \phi : U \to V \), \( p \in U \), so that \( f \circ \phi^{-1} \) has a \underline{pole} at \( \phi(p) \).

3) We say \( f \) has an \underline{essential singularity} at \( p \) if there is a chart \( \phi : U \to V \), \( p \in U \), so that \( f \circ \phi^{-1} \) has an \underline{essential singularity} at \( \phi(p) \).

**Lemma 2.2:** The definitions above are independent of the chart used.

**Proof:** Exercise.

**Def:** We say a function \( f \) on \( X \) is \underline{meromorphic} at \( p \in X \) if it is either holomorphic, has a removable singularity, or a pole at \( p \). We say \( f \) is \underline{meromorphic} on an open set \( U \) if it is meromorphic at every point of \( U \).

It is easy to see if \( f \) and \( g \) are meromorphic at \( p \), then \( f \circ g \), \( f/g \) are meromorphic at \( p \). If \( g \) is not identically \( 0 \) then \( f/g \) is meromorphic at \( p \). This gives that \( M_X \)
defined by

\[ \mathcal{M}_x(U) = \{ f : U \to \mathbb{C} : f \text{ meromorphic} \} \]

is a sheaf of algebras. Moreover, \( \mathcal{O}_x \subset \mathcal{M}_x \) is a subalgebra.

**Example:** Let \( f, g \) be holomorphic at \( p \in X \). Then if \( g \) is not identically 0, \( \frac{f}{g} \) is a meromorphic function at \( p \).

In fact, all meromorphic functions are locally ratios of two holomorphic functions.

**Example:** Let \( X \) be a projective plane curve defined by \( F(x, y, z) = 0 \).

Let \( G \) and \( H \) be homogeneous polynomials of degree \( d \) so that \( H \) does not vanish identically on \( X \). Then \( \frac{G}{H} \) is a meromorphic function on \( X \).

One has Laurent series in this situation as well. The difficulty is that they do depend on the choice of coordinate. However, there are some important properties of the Laurent series that do not depend on the choice of coordinate.
Let $f$ be holomorphic on a punctured disk around $p \in X$. Let

$\phi : U \to V$ be a chart around $p$. We can write $z = \psi(x)$ for $x \in U$,
so $f \circ \phi^{-1}$ is holomorphic in a punctured neighborhood of $z_0 = \phi(p)$. Thus, $f \circ \phi^{-1}$ has a Laurent series

$$f \circ \phi^{-1}(z) = \sum_n c_n (z - z_0)^n.$$  

This is the Laurent series of $f$ around $p$ with respect to $\phi$.

**Def.** Let $f$ be holomorphic at $p$ whose Laurent series with respect to $\phi$ is given by

$$f \circ \phi^{-1}(z) = \sum_n c_n (z - z_0)^n$$  

as above. The **order of $f$ at $p$**, denoted $\text{ord}_p(f)$, is the minimum

exponent occurring in (\#), i.e.,

$$\text{ord}_p(f) = \min \left\{ n : c_n \neq 0 \right\}.$$  

Since the Laurent series depends on $\phi$, one must check this
is well-defined. Let $\psi : U' \to V'$ be another chart around $p$. Set $w = \psi(x)$ for $x \in U'$ and $w_0 = \psi(p)$. Let $T = \phi \circ \psi^{-1}$. Note this gives $\phi$ as a holomorphic function of $w$. We know $T$ is
inversible at $w_0$, we have $T'(w_0) \neq 0$ (check an exercise).
Thus, the power series for \( T \) is of the form

\[
Z = T(w) = Z_0 + \sum_{n \geq 1} a_n (w - w_0)^n
\]

with \( a_1 \neq 0 \). Suppose now the Laurent series for \( f \) with respect to \( \phi \) is

\[
f \circ \phi^{-1}(z) = C_{n_0} (z - z_0)^{n_0} + \sum_{n > n_0} c_n (z - z_0)^n
\]

with \( C_{n_0} \neq 0 \), i.e., \( \ord_p(f; \phi) = n_0 \). Now we calculate the Laurent series for \( f \) with respect to \( \phi \). To get this, we compare \((1)\) with

\[
Z - Z_0 = \sum_{n \geq 1} a_n (w - w_0)^n
\]

This gives immediately the lowest order nonzero term is \( C_{n_0} \), so \( \ord_p(f; \phi) = n_0 \). This gives that \( \phi \) order is well-defined.

**Lemma 2.3:** Let \( f \) be meromorphic at \( p \). Then \( f \) is holomorphic at \( p \) iff \( \ord_p(f) \geq 0 \). In this case \( f(p) = 0 \) iff \( \ord_p(f) > 0 \).

We have \( f \) has a pole at \( p \) iff \( \ord_p(f) < 0 \). The function \( f \) has neither a zero nor a pole at \( p \) iff \( \ord_p(f) = 0 \).

**Proof:** Exercise. \( \Box \)

We also have the following easy lemma.
Lemma 2.4: Let \( f \) and \( g \) be nonzero meromorphic functions at \( p \in X \).

Then
\[
0) \quad \ord_p (fg) = \ord_p (f) + \ord_p (g)
\]
\[
1) \quad \ord_p (f/g) = \ord_p (f) - \ord_p (g)
\]
\[
2) \quad \ord_p (1/g) = -\ord_p (g)
\]
\[
3) \quad \ord_p (f \pm g) \geq \min \left( \ord_p (f), \ord_p (g) \right)
\]

Proof: Exercise.

Example: Let \( C^\infty \) be the Riemann sphere. Let \( f \) be a complex-valued function defined in a nbh of \( \infty \). Then we see by using the chart \( U_\infty \) that \( f \) is meromorphic at \( \infty \) iff \( f(1/z) \) is meromorphic at \( 0 \). (Recall we write \( \infty = \varphi_\infty (x) \), so when we write \( f(z) \) we are really meaning \( f \circ \varphi_\infty (x) \). We showed before
\[
\varphi_\infty \circ \varphi^{-1}_\infty (z) = \frac{1}{z}, \quad \text{and} \quad \varphi_\infty (\infty) = 0.
\]
Thus, we have
\[
f\left(\frac{1}{z}\right) = f\left(\varphi_\infty \circ \varphi^{-1}_\infty (z)\right) = f(\varphi_\infty (x)), \quad \text{which is \( f \) using the correct chart. Be sure you understand this!}
\]
Thus, any rational function \( f(z) = P(z)/Q(z) \) is meromorphic at \( \infty \), so on all of \( C^\infty \). We can factor \( P(z) \) and \( Q(z) \) into linear factors, i.e., we can write \( f(z) \) uniquely as
\[
f(z) = C \prod (z - \lambda_i)^{e_i}
\]
where \( c \) is a nonzero constant, \( \lambda_i \) are distinct complex numbers, and the \( \lambda_i \) are in \( \mathbb{D} \). One sees that this gives \( \text{ord}_x (f) = c \frac{1}{2\pi i} \log x \) for each \( i \). Moreover, \( \text{ord}_x (f) = 0 \) unless \( x = 0 \) or \( x = \lambda_i \) for some \( i \). At \( x = 0 \), we have \( \text{ord}_x (f) = \deg(g) - \deg(p) = -\sum \epsilon_i \).

( Check this is an exercise! ) Thus, we see

\[ \sum_{x \in X} \text{ord}_x (f) = 0. \]

We will later see this is true in general for meromorphic functions on compact Riemann surfaces.

There are many important results from complex analysis that are purely local results, so they transfer immediately to the setting of Riemann surfaces. We list some of the most important here.

**Theorem 2.5:** Let \( f \) be a meromorphic function defined on an open connected subset \( U \subseteq \mathbb{C} \). If \( f \) is not identically 0, then the zeros and poles of \( f \) form a discrete subset of \( U \).

**Corollary 2.6:** Let \( f \) be a meromorphic function on a compact Riemann surface with \( f \) not identically zero. Then \( f \) has a finite
Theorem 2.7: Let \( f \) and \( g \) be meromorphic functions defined on a connected open set \( U \subseteq \mathbb{C} \). Suppose \( f = g \) on a set \( W \subseteq \mathbb{C} \) which has a limit point in \( U \). Then \( f = g \) on \( U \).

Theorem 2.8: (Maximum Modulus Theorem): Let \( f \) be holomorphic on a connected open set \( U \subseteq \mathbb{C} \). Suppose there is a point \( p \in U \) so that \( |f(z)| \leq |f(p)| \) for all \( z \in U \). Then \( f \) is constant on \( U \).

Corollary 2.9: Let \( X \) be a compact Riemann surface. Suppose \( f \) is holomorphic on \( X \). Then \( f \) is a constant function.

Proof: Since \( f \) is holomorphic, \( |f| \) is a continuous function on a compact set \( X \), so it achieves a maximum. Thus, by Thm 2.8 \( f \) is constant.

Example: We have seen that any rational function \( p(z)/q(z) \) is meromorphic on the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \). We can now show any meromorphic function on \( \mathbb{C} \cup \{ \infty \} \) is a rational function.
function. Let \( f \) be a meromorphic function on \( \mathbb{C}^\infty \). We know \( f \) has finitely many zeros and poles, say \( \lambda_1, \ldots, \lambda_n \) and set \( \mathcal{C}_i = \text{ord}_{\lambda_i}(f) \). Let
\[
\mathcal{R}(z) = \prod_{i} (z - \lambda_i)^{\mathcal{C}_i}.
\]
This function has the same zeros and poles as \( f \) in the complex plane \( \mathbb{C} \), all to the same order. Let \( g(z) = \frac{f(z)}{\mathcal{R}(z)} \).

Then \( g \) is a meromorphic function on \( \mathbb{C} \) with no zeros or poles in \( \mathbb{C} \). Since it is holomorphic on \( \mathbb{C} \), it has a Taylor series
\[
g(z) = \sum_{n=0}^{\infty} \mathcal{C}_n z^n,
\]
which converges on \( \mathbb{C} \). Note \( g \) is meromorphic at \( \infty \).

In terms of the coordinate \( w = \frac{1}{z} \) at \( \infty \), we have
\[
g(w) = \sum_{n=0}^{\infty} \mathcal{C}_n w^{-n}.
\]

For this to be meromorphic at \( w = 0 \) we must have
\[
\mathcal{C}_n = 0 \text{ for large enough } n. \quad \text{Thus, } g \text{ is a polynomial in } \mathbb{C}^\infty. \text{ If } g \text{ is not constant, it will have a zero which is a contradiction. Thus, } f = (\text{constant}) \mathcal{R}(z), \text{ i.e., } f
\]
is a rational function.
Example: Consider $Y = P^1$. For this example we state the result, but
leave the verifications as an exercise (that you should do!).

Let $p(z,w)$ and $q(z,w)$ be homogeneous polynomials of the same
degree with $q$ not identically 0. Then $f(z,w) = p(z,w)/q(z,w)$
defines a meromorphic function on $P^1$. One has that every
homogeneous polynomial in $z, w$ factors into linear factors. Thus,
we can write

$$f(z,w) = \prod_i (b_i z - a_i w)^{e_i}$$

where we can assume the factors are relatively prime.

Lemma 2.10: Every meromorphic function on $P^1$ is a ratio of
homogeneous polynomials in $z$ and $w$ of the same degree.

Proof: Let $f$ be meromorphic on $P^1$ with $f$ not identically 0.
Since $P^1$ is compact we know $f$ has finitely poles and
zeros. We list them as $\{a_i : b_i : 1\}$. Write

$$\text{ord}_{(a_i : b_i)} f = e_i.$$ 

Set

$$g(z,w) = \prod_i (b_i z - a_i w)^{e_i}.$$ 

where we choose $n$ to make $g$ a ratio of homogeneous
polynomials, i.e., $n = -\Sigma e_i$. Then $h = f/g$ has no zeros or
pole, except possibly at $[1:0]$ where $w = 0$.

If $h$ has a pole at $[1:0]$, then $h$ cannot have a
zero at $[1:0]$ so it has no zeroes. Thus, $\frac{1}{h}$ has no
poles. Thus, $\frac{1}{h}$ is constant since $P^1$ is compact and $\frac{1}{h}$
is holomorphic. But $\frac{1}{h}$ has a zero at $[1:0]$ so $\frac{1}{h} = 0$.
This is a contradiction because $\frac{1}{h} = 0$ is not zero.
Thus, $h$ does not have a pole at $[1:0]$. Thus $h$ is holomorphic
on $P^1$ so $h$ is constant.

One uses this result to show if $f$ is any nonconstant
meromorphic function on $P^1$ then

$$\sum_{p \in P^1} \text{ord}_p(h) = 0.$$ 

Example: Let $X \subseteq \mathbb{C}^2$ be a smooth affine plane curve
given by $f(z, w) = 0$. We have seen the coordinate
functions $z$ and $w$ are holomorphic on $X$, so any poly
$g(z, w)$ is holomorphic on $X$. Thus, ratios of polynomials
are meromorphic functions on $X$ as long as the denominator
does not vanish on $X$. Essentially this can only happen if
Theorem 2.11: Let $h$ be a polynomial vanishing on $X$ where

$$X = \overline{\{ (x, y) : f_1(x, y) = 0 \}}.$$ Then $f \mid h$.

We can state this as a proposition.

Prop. 2.12: Let $X$ be a smooth affine plane curve defined by an irreducible nonsingular polynomial $f(z, w) = 0$. Then any ratio of polynomials $p(z, w)/q(z, w)$ is a meromorphic function on $X$ as long as $f$ does not divide $q(z, w)$.

We now turn our attention to projective curves. Let $X$ be given as the zero set of the homogeneous polynomial $F(x, y, z)$. Since $X$ is compact, we cannot take ratios of holomorphic functions to get interesting meromorphic functions because $X$ has no nonconstant holomorphic functions. We can still take ratios though. Let $G$ and $H$ be homogeneous polynomials of the same degree. Then $R(x, y, z) = G(x, y, z)/H(x, y, z)$ defines
a complex-valued function on $\mathbb{P}^2$ away from the zeros of $H$. As above, one has $H$ vanishes identically on $X$ iff $F|H$. Suppose this is not the case. Then we claim $R$ is a

meromorphic function on $X$. To see this, we can check things locally. In particular, we can work in the affine charts of $\mathbb{P}^2$. For example, consider $R(x, y, z)$ on $U_0$. Then

$$R|_{U_0}(x, y, z) = \frac{G(x, y, z)}{H(x, y, z)}.$$ 

Setting $g(x, y, z) = G(x, y, z)$ and $h(x, y, z) = H(x, y, z)$, note that $R|_{U_0}$ is meromorphic on $U_0$. The other charts are done similarly. Thus, we have the following proposition.

**Prop. 2.13:** Let $X$ be a smooth projective plane curve defined by an irreducible non-singular homogeneous polynomial $F(x, y, z)$. Then any ratio of homogeneous $G(x, y, z)/H(x, y, z)$

where $G$ and $H$ have the same degree and $F \neq 0$, define a meromorphic function on $X$.

We have seen plane projective curves. It is not the case
That all interesting projective curves live in $\mathbb{P}^2$.

**Def.** Let $X \subseteq \mathbb{P}^n$ be a Riemann surface. We say $X$ is **holomorphically embedded** in $\mathbb{P}^n$ if for every $p \in X$ there is a homogeneous coordinate $z_j$ s.t.

1. $z_j \neq 0$ at $p$
2. for every $k$, the ratio $z_k/z_j$ is a holomorphic function at $p$
3. there is a homogeneous coordinate $z_i$ so that $z_i/z_j$ is a local coordinate on $X$ near $p$.

We call such an $X$ a **smooth projective curve**.

**Prop. 2.14:** Let $X$ be a smooth projective curve in $\mathbb{P}^n$. Then any ratio of homogeneous polynomials $\frac{G(z_0,\ldots,z_n)}{H(z_0,\ldots,z_n)}$, where $G$ and $H$ have the same degree define a meromorphic function on $X$ as long as $H$ does not vanish identically on $X$.

**Exercise:** 1) Show $\mathbb{P}^1$ is a smooth projective plane curve.
2) Any smooth projective plane curve $X \subseteq \mathbb{P}^2$ is a smooth projective plane curve.
Now that we have a good notion of what a Riemann surface is, the next step is to define appropriate maps between them. We let $X$ and $Y$ be Riemann surfaces.

**Def.** A map $F: X \rightarrow Y$ is holomorphic at $p \in X$ iff there are charts $\phi_1: U_1 \rightarrow V_1$ on $X$ with $p \in U_1$, and $\phi_2: U_2 \rightarrow V_2$ on $Y$ with $F(p) \in U_2$ so that $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$. If $W \subseteq X$ is open, we say $F$ is holomorphic on $W$ if it is holomorphic at each $p \in W$. We say $F$ is holomorphic if it is holomorphic on $X$.

**Exercise:** Check this definition does not depend on the choice of charts.
Lemma 2.15: We have the following properties of holomorphic maps.

1) The composition of holomorphic maps is holomorphic.

2) The composition of a holomorphic map \( F: X \rightarrow Y \) and a holomorphic function \( g: W \rightarrow \mathbb{C}, \quad W \subseteq Y, \)
   is a holomorphic function on \( F^{-1}(W) \).

3) Let \( F: X \rightarrow Y \) be a holomorphic map and \( g \) a holomorphic function on an open set \( W \subseteq Y \). Assume \( F(X) \) is
   not a subset of the set of poles of \( g \). Then \( g \circ F \)
   is meromorphic on \( F^{-1}(W) \).

Proof: Exercise. 

We can rephrase the last two properties in terms of sheaves. Let \( F: X \rightarrow Y \) and let \( \mathcal{O}_Y \) be a sheaf on \( Y \).

We define a sheaf \( F_* \mathcal{O}_Y \) on \( Y \) by setting, for each
open set \( W \subseteq Y, \quad F_* \mathcal{O}_Y(W) = \mathcal{O}_Y(F^{-1}(W)). \) Note \( F^{-1}(W) \)
open in \( X \), in this manner sense. The sheaf \( F_* \mathcal{O}_Y \) is called
the pushforward of \( \mathcal{O}_Y \). In our case, we consider the sheaf
\( F_* \mathcal{O}_X \). There is a natural map of sheaves
\[ F^\#: \mathcal{O}_Y \to F_* \mathcal{O}_X, \]

For each \( W \subseteq Y \) there is a \( \mathcal{O}_X \)-algebra homomorphism
\[ F^\#: \mathcal{O}_Y (W) \to \mathcal{O}_X (F^{-1}(W)). \]

This map is given by \( F^\#(g) = g \circ F \). One has the same result for the sheaves \( \mathcal{M}_Y \) and \( F_* \mathcal{M}_X \).

**Exercise:** Show that if \( F: X \to Y \) and \( G: Y \to Z \), then
\[ (G \circ F)^\# = F^\# \circ G^\#. \]

**Definition:** We say Riemann surfaces \( X \) and \( Y \) are isomorphic (or holomorphic) if there is a bijective holomorphic map \( F: X \to Y \) so that \( F^{-1}: Y \to X \) is also holomorphic.

**Example:** \( \mathbb{C} \) and \( \mathbb{P}^1 \) are isomorphic. The map is given by
\[ [z: w] \mapsto \frac{1}{1 + w^2} \begin{pmatrix} 2 \Re(z \overline{w}) & 2 \Im(z \overline{w}) \end{pmatrix}, 1z^2 + 1w^2). \]

As before, we can translate some results from complex analysis to the setting of Riemann surfaces. Recall the open mapping theorem.
from complex analysis. This translates to Riemann surfaces.

**Theorem 2.16:** Let \( F : X \to Y \) be a non-constant holomorphic map between Riemann surfaces. Then \( F \) is an open mapping.

Note this is one of the many theorems that distinguish between holomorphic and real differentiability. For example, consider \( f(x) = x^2 \) on \( \mathbb{R} \).

**Prop. 2.17:** Let \( F : X \to Y \) be a 1-1 holomorphic map. Then \( F \) is an isomorphism between \( X \) and \( F(X) \), i.e.,

\[
F^{-1} : F(X) \to X \text{ in holomorphic.}
\]

**Prop. 2.18:** Let \( F \) and \( G \) be holomorphic maps between Riemann surfaces \( X \) and \( Y \). If \( F = G \) on a subset \( S \subseteq X \) with a limit point in \( X \), then \( F = G \).

Both of the previous results follow immediately from the analogous results from complex analysis. The next result deals with compact Riemann surfaces.

**Prop. 2.19:** Let \( X \) be a compact Riemann surface and let...
F: X → Y be a nonconstant holomorphic map. Then Y is compact and F is onto.

Proof: Since F is holomorphic, it is open, so F(x) is open.

However, since X is compact and F is continuous, F(x) is compact. Now since Y is Hausdorff, this gives F(x) is closed in Y. However, now we use the Y is connected and F(x) is open and closed to conclude Y = F(x).

Prop 2.70 (Discreteness of Preimages): Let F: X → Y be a nonconstant holomorphic map between Riemann surfaces. For every y ∈ Y, the preimage F⁻¹(y) is a discrete subset of X.

In particular, if X is compact then F⁻¹(y) is a nonemptily finite set for every y ∈ Y.

Proof: Fix a chart φ: U → V centered at y, i.e., φ(y) = 0.
Write this in local coordinates as z, i.e., for x ∈ U, write z = φ(x). Let x ∈ F⁻¹(y) and choose a chart ψ: U' → V' centered at x. Write the local coordinate as w. Then we can express F in terms of w.
and \( z \) as a holomorphic function \( g(w) = z \), i.e.,

\[ g = \phi \circ F \circ \psi^{-1}. \]

Note since the charts are centered at \( x \) and \( y \) we have \( g(a) = 0 \). We know zeros of holomorphic functions on \( C \) are discrete, then \( g(w) = 0 \) so that \( g(w) \) is \( w \neq 0 \). Translated, this means there are no preimages of \( y \) in \( \psi^{-1}(\psi') \). Thus, the preimages of \( y \) are discrete. The rest follows from Prop 2.19, and the fact that discrete subsets of compact sets are finite.

It turns out that any nonconstant holomorphic function can be represented very simply for the correct choice of coordinate, namely, it will just be a power map.

**Prop 2.21 (Local Normal Form):** Let \( F : X \rightarrow Y \) be a holomorphic map defined at \( p \in X \). There is a unique \( m \in \mathbb{Z}_{\geq 1} \)

which satisfies: for every chart \( \phi_2 : U_2 \rightarrow V_2 \) on \( Y \) centered at \( F(p) \), there is a chart \( \phi_1 : U_1 \rightarrow V_1 \) centered at \( p \) so that \( \phi_2(F(\phi_1^{-1}(2))) = Z^m \).

**Proof:** Fix a chart \( \phi_1 : U_1 \rightarrow V_1 \) on \( Y \) centered at \( F(p) \).
Let \( \phi : U \rightarrow V \) be any chart centered at \( p \). Set \( T(w) = \phi_2 \left( F(\phi^{-1}(w)) \right) \). The Taylor series for \( T \) can be written as \( T(w) = \sum_{i=m}^{\infty} c_i w^i \) with \( c_m \neq 0 \) for some \( m > 1 \) since \( T(0) = 0 \). Thus, we can write

\[
T(w) = w^m S(w)
\]

for \( S \) a holomorphic function at \( w = 0 \) with \( S(0) \neq 0 \). Thus, we can find a holomorphic function \( R(w) \) near 0 so that \( R(w)^m = S(w) \). Thus, \( T(w) = (w R(w))^m \). Let \( \eta(w) = w R(w) \).

We have \( \eta'(w) = R(w) + w R'(w) \) so \( \eta'(0) = R(0) \neq 0 \). Thus, the implicit function theorem says \( \eta \) is invertible near 0. Let \( \phi_1 = \eta \circ \phi \). Then \( \phi_1 \) is a chart on \( X \) defined and centered at \( p \). Let \( Z = \eta(w) \), then \( Z = w R(w) \).

Thus,

\[
\phi_2 \left( F(\phi^{-1}(Z)) \right) = \phi_2 \left( F(\phi^{-1}(\eta^{-1}(Z))) \right)
\]

\[
= T(\eta^{-1}(Z))
\]

\[
= T(w)
\]

\[
= (w R(w))^m
\]

\[
= Z^m.
\]

It only remains to prove that \( m \) is unique.
Since the local coordinates give near $p$ that $F$ (in coordinates) is $Z \to Z^m$, we see each point near $F(p)$ has $m$ preimage points near. Thus, $m$ is given by preimage points, so it cannot vary with chart.

**Def:** The multiplicity of $F$ at $p$, denoted $\text{mult}(F)$, is the unique integer $m$ such that there are local coordinate near $p$ and $F(p)$ with $F$ having the form $Z \to Z^m$.

We always have $\text{mult}(F) \geq 1$. This is clear when we think in terms of preimages.

It is generally worthwhile to have to find local coordinates to determine the multiplicity. Let $Z$ be any local coordinate near $p$ and $\omega$ any local coordinate near $F(p)$. Say $\phi(p) = \omega_0$ and $\psi(F(p)) = \omega_0$. So we can write $F$ as

$w = h(z)$ in these coordinates.

**Lemma 2.22:** With notation as above, we have

$$\text{mult}(F) = 1 + \text{ord}_{\omega_0} \left( \frac{dh}{dz} \right)$$
When $\text{ord}_{z_0}(f)$ is the order of vanishing of $f$ at $z_0$. In particular, if $h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i (z - z_0)^i$
with $m \geq 1$ and $c_m \neq 0$, then $\text{mult}(F) = m$.

**Proof:** Exercise. This is essentially the local normal form, just shift the coordinates from $0$ to $z_0$ and work.

**Def:** Let $F: x \mapsto y$ be a meromorphic holomorphic map. A point $p \in X$ is a *ramification point* for $F$ if $\text{mult}(F) \geq 2$.

A point $y \in Y$ is a *branch point* for $F$ if it is the image of a ramification point for $F$.

We again illustrate with a picture that is not in the correct dimension, but gives an idea.
Note we have three ways to compute $\text{mult}(F)$; the m that occurs as the local monodromy from the mumber of preimage points, or the formula in Lemma 2.22.

**Example:** Let $X$ be a smooth affine plane curve given by $F(z, w) = 0$.

Define $F: X \rightarrow \mathbb{C}$ by projection onto the $z$-coordinate.

Let $p = (z_0, w_0) \in X$.

Assume $\frac{2F}{\partial z}(p) \neq 0$. Then $F$ is a chain for $X$ near $p$. Since $F$ is the logarithm near $p$, clearly, we have $\text{mult}(F) = 1$.

Now assume $\frac{2F}{\partial z}(p) = 0$. Then since $X$ is smooth, we have $\frac{2F}{\partial z}(p) \neq 0$. Thus, the implicit function theorem gives that near $p$, $X$ is the graph of a holomorphic function $z = g(w)$. Thus, near $p$ we have $F(g(w), w) = 0$. We differentiate with respect to $w$ to obtain

$$0 = \frac{2F}{\partial z} \frac{2g}{\partial w} + \frac{2F}{\partial w}.$$ 

Since $\frac{2F}{\partial w}(p_1) = 0$, we have $\frac{2F}{\partial z}(p) \frac{2g}{\partial w}(p) = 0$.

We have assumed $\frac{2F}{\partial z}(p_1) \neq 0$, so $\frac{2g}{\partial w}(p_1) = 0$. Thus, Lemma 2.22 gives that $\text{mult}(F) \geq 2$.

Summarizing, we see $F$ is ramified at $p$ iff
Example: We now give a more specific example. Let $X = Y = \mathbb{C}$ and consider $F(x) = x^2$. We calculate the multiplicity of this point in several ways. First, Lemma 2.21 answers this very easily because $F$ is globally given by such an $h$. Thus, for every $p \in X$ we have

$$\text{mult}_p(F) = 1 + o_\nu p(2x).$$

Thus, if $p = 0$ then $\text{mult}_p F = 1$ and if $p \neq 0$ we have $\text{mult}_p(F) = 2$.

Now let's write $F$ in local normal form. Clearly, $x = 0$ is already in local normal form. Thus, we only need to compute there. Let $p \in X$ with $p \neq 0$. Our chart on $Y$ near $F(p)$ is given by $\phi_1(y) = y - p^2$ and the chart near $p$ in $\Psi(x) = x - p$. Thus, we have

$$T(w) = \phi_1 \circ F \circ \Psi^{-1}(w) = \phi_2 \circ F(w + p)$$

$$= \phi_2((w + p)^2)$$

$$= (w + p)^2 - p^2$$

$$= w^2 + 2wp .$$

Thus,

$$T(w) = wS(w)$$

when $S(w) = w + 2p$ as long as $p \neq 0$. Hence $m = 1$. 

(67)
there is no need to introduce the function $R$ here.

Let $Z = w S(w) = w^2 + 2wp$. Note that $T'(w) = 2w + 2p$, so $T'(0) \neq 0$. Thus, $\varphi$ is invertible. This gives

$$
\varphi_1 = T \circ \varphi$$

is a chart on $X$ near $p$. Thus,

$$
\varphi_2 \circ F \circ \varphi_1^{-1}(z) = T(T^{-1}(z)) = z.
$$

This shows the local normal form near $p \neq 0$ is $Z \to Z$, so the multiplicity is 1.

Finally, the last way is in terms of counting preimage points. Consider $y$ near $F(p)$. Then there are precisely $\text{mult}(F)$ points in $F^{-1}(y)$ near $p$. For example, given $p = 1$, $F(p) = 1$ as well. Now for any point $y$ near $1$ we have two points in $F^{-1}(y)$, but only one will be close to 1 as the other will be close to -1. Thus, $\text{mult}(F) = 1$. For a point near zero, the positive and negative $y$-roots will both be near 0 as $\text{mult}(F) = 2$. This method is useful for figuring out the multiplicities, but not proving it.

**Exercise:** Show $\exp : C \to C$ has no ramification points.

**Exercise:** Let $F : X \to Y$. Show that if $w = F(z)$ is a local representation of $F^p$ near $p$, then
the ramification points of $F$ near $p$ are the zeros of $F'(z)$.

2) Use part (1) to show the points with $\text{mult}(F) \geq 2$ form a discrete subset of $X$.

We can relate the notion of order for a meromorphic function to multiplicity as well. Let $f: X \to \mathbb{C}$ be a holomorphic function. We can view this as a holomorphic map since clearly $\mathbb{C}$ is a Riemann surface. Now suppose $f$ is a meromorphic function.

We want to define the value of $f$ at a pole to be $\infty$. Thus, define $F: X \to \mathbb{C}_\infty$ by

$$F(x) = \begin{cases} f(x), & x \text{ is not a pole of } f \smallskip \\ \infty, & x \text{ is a pole of } f. \end{cases}$$

Then $F$ is a holomorphic map. In fact, we have a bijection between meromorphic functions on $X$ and holomorphic functions $F: X \to \mathbb{C}_\infty$ that are not identically $\infty$.

Now let $f$ be a meromorphic function on $X$ and $F: X \to \mathbb{C}_\infty$ the associated holomorphic map. Suppose $p \in X$ is not a pole of $f$. Set $z_0 = f(p)$. Then $f - z_0$ has a zero at $p$ and by Lemma 2.22 we see $\text{mult}(F) = \text{ord}_p(f - f(p))$. 

(End)
Now suppose \( p \) is a pole of \( f \) so \( \operatorname{ord}_p(f) < 0 \), i.e., \( p \) is a zero of \( \frac{1}{f} \). Now we use the same argument to see
\[
\operatorname{mult}_p(f) = -\operatorname{ord}_p(f). \quad \text{Thus, we have}
\]

**Lemma 2.23:** Let \( f \) be a meromorphic function on \( X \) with associated holomorphic map \( F : X \rightarrow \mathbb{C}^n \).

1) if \( p \in X \) is a zero of \( f \), then \( \operatorname{mult}_p(f) = \operatorname{ord}_p(f) \)

2) if \( p \) is a pole of \( f \), then \( \operatorname{mult}_p(f) = -\operatorname{ord}_p(f) \)

3) if \( p \) is neither a zero nor pole of \( f \), then
\[
\operatorname{mult}_p(f) = \operatorname{ord}_p(f-\bar{f}(p)).
\]

The following result is a very important property of holomorphic maps between compact Riemann surfaces.

**Theorem 2.24:** Let \( F : X \rightarrow Y \) be a nonconstant holomorphic map between compact Riemann surfaces. For each \( y \in Y \), define
\[
d_y(F) = \sum_{p \in F^{-1}(y)} \operatorname{mult}_p(F).
\]

Then \( d_y(F) \) is constant.

**Proof:** We will show that the map \( y \mapsto d_y(F) \)

from \( Y \) to \( \mathbb{Z} \) is locally constant. Since \( Y \) is connected
and $Z$ discrete, this gives the map is constant.

We first consider a side result. Let $f: D \to D$ be given by $z \mapsto z^m$ when $D = \{ z \in \mathbb{C} : |z| < 1 \}$. The map $f$ is holomorphic and onto. Moreover, the only ramification point is at $z=0$. We have $\text{mult}_0(f) = m$ and $\text{mult}_{1/z}(f) = \frac{m}{2}$ for $z \neq 0$.

For $w \in D$, $w \neq 0$, there are $m$ preimage points, each of multiplicity 1 and $\text{mult}_0(f) = m$, so this map satisfies $g_0(f) = m$ for all $y \in D$. Moreover, if one has a disjoint union of such maps, we have the overall function is locally constant.

We now return to our situation.

Fix $y \in Y$ and let $F^{-1}(y) = \{ x_1, \ldots, x_n \}$. Choose local coordinate $w$ near $y$. For each $i$, we can choose local coordinate $z_i$ near $x_i$ so that near $x_i$ we have $F(z_i) = z_i^m$. This gives the desired disjoint union description of $F$. It remains to prove that near $y$ there are no preimages unaccounted for which are not in the orbits of the $x_i$. We will now make use of the compactness of $X$.

Suppose that arbitrarily close to $y$ there are preimages not in any of the orbits of the $x_i$. We can then find
a sequence in \( X \), more of the \( n \) to the right of the \( x \): so that
the image of the sequence converges to \( y \). Since \( X \) is compact
there is a convergent subsequence \( \{ p_n \} \) that converges to a point
in \( X \) and \( F(p_n) \to y \). However, since \( F \) is continuous
we have \( F(p) = y \). This is a contradiction. Thus, there are
no uncounted preimages in a neighborhood of \( y \). This gives
\( y = \delta_y(F) \) is locally constant as claimed.

\[ \text{Def:} \quad \delta_y : X \to \mathbb{Y} \text{ be a non-constant holomorphic map between}
\text{compact Riemann surfaces. The degree of } F, \text{ denoted } \deg(F),
\text{ is } \delta_y(F) \text{ for any } y \in Y. \]

\[ \text{Corol. 2.25:} \quad \text{A holomorphic map between compact Riemann}
\text{surfaces is an isomorphism iff it has degree 1.} \]

\[ \text{Prop. 2.26:} \quad \text{Let } X \text{ be a compact Riemann surface having a}
\text{meromorphic function } f \text{ with a single simple pole. Then}
X \text{ is isomorphic to } \mathbb{C} \! \setminus \! \{ 0 \}. \]

\[ \text{Proof:} \quad \text{Let } f \text{ be a meromorphic function on } X \text{ with a}
\text{single simple pole at } p. \text{ Then the associated map} \]
F: X \rightarrow \mathbb{C} \text{ satisfies } \text{mult}_p(F) = 1 \text{, since } p \text{ is the only point mapping to } \infty, \text{ we have } \deg F = 1. \text{ Thus } F \text{ is an isomorphism.}

We are now ready to prove a result we stated before and proved in the single case } X = \mathbb{C}.

\textbf{Thm 2.27:} Let } f \text{ be a non-constant meromorphic function on a compact Riemann surface } X. \text{ Then } \sum_{p \in X} \text{ord}_p f = 0.

\textbf{Proof:} Let } F: X \rightarrow \mathbb{C} \text{ be the associated holomorphic map. Set } Z = \{x_i \} = F^{-1}(0) \text{ and } Z = F^{-1}(\infty). \text{ In particular, } X \text{ is the zeros of } f \text{ and the } y_j \text{ are the poles of } F.

Let } d = \deg F. \text{ Then we have}

\[ d = \sum_{i} \text{mult}_{x_i}(F) = \sum_{j} \text{mult}_{y_j}(F). \]

The only points where } \text{ord}_p(f) \neq 0 \text{ are amongst the } x_i \text{ and } y_j. \text{ Moreover, we have}

\[ \text{mult}_{x_i}(F) = \text{ord}_{x_i}(F) \]

\[ \text{and } \text{mult}_{y_j}(F) = -\text{ord}_{y_j}(F). \]

Thus, we have
\[
\sum_{\text{ord}_x(F)} = \sum_{i} \text{ord}_x(F_i) + \sum_{j} \text{ord}_y(F_j)
\]

\[
= \sum \text{mult}_x(F) - \sum \text{mult}_y(F)
\]

\[
= d - d = 0.
\]

We end this chapter with one of the more useful formulas: the Hurwitz formula. Unfortunately, the set-up requires some topology since we need to introduce the topological genus. Here we will give a more thorough introduction to the arithmetic and analytic genus. It will turn out they are all the same. The following is a fundamental result to the theory.

**Theorem 2.28:** Every compact Riemann surface is diffeomorphic to a \(g\)-hled torus for some unique integer \(g \geq 0\).

The integer \(g\) is called the topological genus of the Riemann surface. One can also compute this in terms of triangulations of \(X\).

**Def:** A triangulation of \(X\) is a decomposition of \(X\) into closed subsets, each homeomorphic to a triangle, such that any two triangles are either disjoint, meet at only
DEF: Let \( X \) be a compact Riemann surface. Suppose we have a triangulation of \( X \) with \( v \) vertices, \( e \) edges, and \( t \) triangles. Then the Euler number of \( X \) with respect to this triangulation is the integer

\[ X(X) = v - e + t. \]

Prop. 2.27: For a Riemann surface \( X \), of topological genus \( g \), \( X \) is triangulable and \( X(X) = 2 - 2g. \)

Example: 1) Let \( X = \mathbb{C}P^2 \). Then we can triangulate as follows:

There are 6 triangles, 9 edges, and 5 vertices.

Thus, \( X(\mathbb{C}P^2) = 5 - 9 + 6 = 2 \). This gives

\[ 2 = 2 - 2g, \quad \text{i.e.,} \quad g = 0. \]

2) Let \( X \) be the torus. Then we can most easily draw a triangulation as follows:
There are 2 triangles, 3 edges, and 1 vertex. Thus,
\[ \chi(X) = 1 - 3 + 2 = 0. \] This gives \( 2 - 2g = 0 \), i.e., \( g = 1 \).

**Theorem 2.30 (Hurwitz's Formula):** Let \( F : X \to Y \) be a non-constant holomorphic map between compact Riemann surfaces.

Then

\[ 2g(X) - 2 = \deg(F) \left( 2g(Y) - 2 \right) + \sum_{p \in X} \left( \text{mult}(F, p) - 1 \right). \]

**Proof:** We first note that \( X \) is compact, so the set of ramification points is finite. We have \( \text{mult}(F, p) - 1 = 0 \) unless \( p \) is a ramification point. Thus, the sum is really a finite sum over ramification points of \( F \).

Consider a triangulation of \( Y \) where each branch point of \( F \) is a vertex. Suppose we have \( v \) vertices, \( t \) triangles, and \( e \) edges. We can lift this triangulation to \( X \) via \( F \). Assume there are \( v', t', \) and \( e' \) vertices, triangles and edges on \( X \). Note by definition every ramification point is a vertex in \( X \).

Note there are no ramification points in the triangles.
So each triangle in $Y$ lifts to $degF$ triangles in $X$.

This gives $t' = deg(F) t$. Similarly, we get $e' = deg(F) e$.

It remains to deal with the vertices. Let $q \in Y$ be a vertex. The number of preimages of $q$ is given by $|F^{-1}(q)|$. We write this as

$$|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1$$

$$= deg F + \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F)).$$

Thus, we have

$$V' = \sum_{\text{vertex } q \text{ of } Y} \left( deg F + \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F)) \right)$$

$$= deg(F) V - \sum_{\text{vertex } q \text{ of } Y} \sum_{p \in F^{-1}(q)} (\text{mult}_p(F) - 1)$$

$$= deg(F) V - \sum_{p \text{ vertex of } X} (\text{mult}_p(F) - 1).$$

Therefore, we have

$$2g(X) - 2 = -\chi(X)$$

$$= -V' + e' - t'$$

(67)
\[-\text{deg}(F) \chi + \sum_{\text{vertex } p \text{ of } X} (\text{mult}_p(F) - 1) + \text{deg}(F) e - \text{deg}(F) t \]

\[-\text{deg}(F) \chi + \sum_{\text{vertex } p \text{ of } X} (\text{mult}_p(F) - 1) \]

\[-\text{deg}(F) \chi + \sum_{\text{vertex } p \text{ of } X} (\text{mult}_p(F) - 1) \]

Since the term in the last sum is 0 unless \( p \) is a ramification point and all the ramification points are among the vertices in \( X \).

\[\Box\]

We now conclude the chapter with one more example. We will make use of Hurwitz's formula in the example. We begin by gluing together Riemann surfaces. Note that given a Riemann surface, each open set \( U \subseteq X \) is a Riemann surface as well.

In this regard one can think of a Riemann surface as composed of a bunch of Riemann surfaces stuck together. However, given any Riemann surface \( X \) and \( Y \) sometimes we can glue them together.
Let $X$ and $Y$ be Riemann surfaces and suppose there are open sets $U \subseteq X$, $V \subseteq Y$ and $\phi: U \to V$ an isomorphism.

Consider the disjoint union $X \sqcup Y$ and define an equivalence relation by defining equivalence classes $\equiv$: $(x, x') \equiv (y, y')$ if $x \in U$, $y \in V$, and $(x, \phi(x)) \equiv (y, \phi^{-1}(y))$. We denote the resulting space by $X \sqcup Y/\phi$.

**Prop. 2.31:** There is a unique complex structure on $X \sqcup Y/\phi$ so that the natural inclusions $X \hookrightarrow X \sqcup Y/\phi$ and $Y \hookrightarrow X \sqcup Y/\phi$ are holomorphic. Moreover, if $X \sqcup Y/\phi$ is Hausdorff, it is a Riemann surface.

**Proof:** Just take the union of an atlas for $X$ and an atlas for $Y$. The charts are compatible because $\phi$ is an isomorphism.

**Exercise:** Let $X = \mathbb{C}$, $Y = \mathbb{C}$.

a) Show that $U = \mathbb{C}^x$, $V = \mathbb{C}^y$ and $\phi(x) = x^2$ then $X \sqcup Y/\phi \cong \mathbb{P}^1$. 

(69)
b) Show that $U = \mathbb{C}^x, V = \mathbb{C}^x$, and if $|z| > 2$ then $U \times \mathbb{C} / \phi$ is not Hausdorff.

The main example we would like to study here is that of a hyperelliptic surface. Let $h(x) \in \mathbb{C}[x]$ with $\deg h = 2g + 1 + \varepsilon$ where $\varepsilon = 0$ or $1$.

Set $K(x) = x^{2g+2} h(\frac{1}{x}) \in \mathbb{C}[x^2]$. If $g > 0$ we take $\varepsilon = 1$ so $\deg h > 2$. We assume $h$ has only simple roots (no it does as well.) Consider the smooth affine plane curves

$$X_h = \{ (x,y) : y^2 = h(x) \}$$

and

$$X_k = \{ (x,y) : y^2 = k(x) \}.$$

Take the open sets

$$U_h = \{ (x,y) \in X_h : x \neq 0 \}$$

and

$$U_k = \{ (x,y) \in X_k : x \neq 0 \}.$$

We define an isomorphism $\phi : U_h \rightarrow U_k$ by setting

$$\phi(x,y) = \left( \frac{1}{x}, \frac{y}{x^{g+1}} \right).$$
Lemma 2.32: The set $Z$ is a compact Riemann surface of genus $g$. The meromorphic function given by projection to the $x$-coordinate from $X_h$ extends to a meromorphic map $\pi : Z \rightarrow \mathbb{P}^1$ of degree $2$. Every root of $h$ is a branch point of $\pi$, and there is at most one other branch point. There is an other branch point iff $e=0, \quad d \geq 2$.

Proof: We start with compactness. We claim $Z$ is the union of the images of the compact sets $\tilde{S} (x,y) : X_h : |x| \leq 1$ and $\tilde{S} (z,w) : X_h : |z| \leq 1$, as is a compact set. Note if $(x,y) \in X_h$ with $|x| > 1$, then $(x,y) \in U_h$. Thus, it can be identified with $\left( \frac{1}{x}, \frac{y}{x^2} \right) \in U_h$. But then $\vert \frac{1}{x} \vert > 1$, so it is in $\tilde{S} (z,w) : X_h : |z| \leq 1$. This gives the result.

Next we must show $Z$ is Hausdorff. Any two points of $X_h$ that lie in $U_h$ can be separated in $Z$ because they can be separated in $U_h$. The same argument works for $U_h$. Consider a point $p \in X_h$ with $x$-coordinate 0. Let $g \in Z$ be any other point. All $q \in X_h$, we are fin.
Suppose \( q \in X_h \). We take a nbhd of \( p \) in \( X_h \) small enough so the \( x \)-coordinate under \( q \) is very large.

We can choose this so they miss \( y \). Thus, \( p \) and \( q \) can be separated and so \( Z \) is Hausdorff. Thus, \( Z \) is a compact Riemann surface.

Consider the complement of \( X_h \) in \( Z \). It has either 2 or 2 points depending on the order of \( h \) at \( \infty \). We want to extend \( \pi : (x,y) \mapsto x \) to a monomorvhic function on \( Z \). On \( X_h \) define \( \pi_h : (x,y) \mapsto \frac{1}{x} \). This agrees with \( \pi \) on \( \Phi(U_h) \) because

\[
\pi_h(f(x,y)) = \pi_h(\frac{1}{x}, \frac{y}{x^3}) = \frac{1}{x} = x = \pi_h(x,y).
\]

Thus, we have extended the prop to \( Z \). (This also could have been done by looking at \( \pi : Z - Y \) defined by

\[
\pi(x,y) = \begin{cases} F(x) & (x,y) \in X_h \\ \frac{1}{x} & (x,y) \in X_h.
\end{cases}
\]

The degree of \( \pi \) is clear because if \( x \) is not a root of \( h \), the previous points are \( (x, \pm \sqrt{h(x)}) \). The

nutation point occur when \( h(x) = 0 \) or \( x = 0 \),

at \( \infty \) because \( \pi(\infty) \) is not a root of \( h(x) \) iff \( \deg x + \deg h \).
This gives in Hurwitz's formula

\[ \sum_{\text{primes } p \mid 13} (\text{mult}(p) - 1) = 2 \gamma + 2. \]

Thus, we have

\[ 2 \gamma(2) - 2 = \text{deg}(F)(2 \gamma(\mathbb{P}^1) - 2) + (2 \gamma + 2) \]

\[ = 2(-2) + 2 \gamma + 2 \]

\[ = 2 \gamma - 2. \]

Thus, \( \gamma(2) = 5. \)

Def: A Riemann surface \( X \) that gives a double cover

\[ F: X \rightarrow \mathbb{P}^1 \]

is called hyperelliptic. The map \( F \) is called a hyperelliptic projection.

Fact: Every hyperelliptic surface is given as \( \pi \circ \sigma = \pi^\circ \sigma \) for some \( h. \)

Let \( Z \) be defined by \( y^2 = h(x) \). Then \( Z \) has an automorphism \( \sigma: Z \rightarrow Z \) given by \( \sigma(x, y) = (x, -y) \). In fact, \( \sigma \) is an involution. We have \( \pi \circ \sigma = \pi. \)
Lemma 2.34: Every meromorphic function $f$ on a hyperelliptic Riemann surface $X$ defined by $y^2 = h(x)$ can be written uniquely as

$$f(x) = r(x) + y s(x)$$

where $r$ and $s$ are rational functions of $x$.

Proof: Exercise. It basically follows from using the involution $\sigma$ and pulling meromorphic function back from $\mathbb{P}^1$. The proof is also in the textbook.