As was stated in Chapter 0, integration along contours is an incredibly powerful tool in complex analysis. It is natural to extend this notion to Riemann surfaces. We begin with some basics from $C$.

**Def.** A holomorphic 1-form on an open set $V \subset C$ is an expression $\omega = f(z)dz$ for $f$ a holomorphic function on $V$. We say $\omega$ is a holomorphic 1-form in the coordinate $z$.

This is a local definition, but to be able to integrate along a contour we need to be able to move along a curve. This means we need a definition of what 1-forms are compatible on overlapping sets.

**Def.** Let $\omega_1 = f_1(z)dz$ be a holomorphic 1-form on $V_1$, and $\omega_2 = f_2(w)dw$ a holomorphic 1-form on $V_2$. Let $z = T(w)$ be a holomorphic map $V_2 \to V_1$. We say $\omega_1$ transforms to $\omega_2$ under $T$ if $g(w) = f(T(w))T'(w)$. 
Note how we are getting this transformation law by essentially requiring that \(dz = T(w) dw\).

**Def:** Let \(X\) be a Riemann surface. A **holomorphic 1-form on** \(X\) is a collection of holomorphic 1-forms \(\omega_j\), one for each chart \(\varphi : U \to V\) in the coordinate of \(V\), so that if two charts \(\varphi_1 : U_1 \to V_1\) have overlapping domain, then the associated holomorphic 1-form \(\omega_1\) transforms to \(\omega_2\) under the change of coordinates mapping \(T = \varphi_1 \circ \varphi_2^{-1}\).

One doesn't really need a form for each chart on \(X\), it is enough just to have compatible 1-forms on an atlas on \(X\).

**Prop. 3.1:** Let \(X\) be a Riemann surface and \(A\) an atlas on \(X\).

Assume for each chart in \(A\) we have a holomorphic 1-form.

Moreover, assume these charts are compatible. Then there exists a unique holomorphic 1-form on \(X\) extending these holomorphic 1-forms on each chart of \(X\).

**Proof:** Exercise.
For a holomorphic 1-form we used \( w = f(z) \, dz \) for a while. It is natural to extend this to meromorphic functions.

**Def:** A **meromorphic 1-form** on an open set \( V \subseteq \mathbb{C} \) is an expression \( w = f(z) \, dz \) where \( f \) is a meromorphic function on \( V \). We say \( w \) is a **meromorphic 1-form** in \( V \).

Our now proceeds exactly in the holomorphic case to define meromorphic forms on Riemann surfaces and obtain the analogous result to Prop. 3.1.

Let \( w \) be a meromorphic 1-form in a neighborhood \( U \) of \( p \in X \).

Choose local coordinates centered at \( p \). Then we can write

\[
  w = f(z) \, dz
\]

where \( f \) is a meromorphic function at \( z = 0 \). We write \( \text{ord}_p(w) \) for \( \text{ord}_p(f) \).

**Exercise:** Show \( \text{ord}_p(w) \) is well-defined.

From this we see that \( w \) is holomorphic at \( p \) iff \( \text{ord}_p(w) = 0 \).
As in the case of functions, we say \( p \) is a zero of \( w \) of order \( n \), \( \text{ord}_p(w) = n > 0 \), \( \text{ord}_p(w) = -n < 0 \). From our previous results we see the set of zeroes and poles of a meromorphic 1-form form a discrete set.

Note that often one can define a 1-form with a single formula \( w = f(z)dz \) on some chart \( p: U \rightarrow V \). The reason is that we have seen that if two meromorphic functions agree on an open set that they are the same. One can often use this to extend \( w \) from \( U \) to all of \( X \). Of course one has to be careful. For example, \( w = \exp(1/z)dz \) is a meromorphic 1-form on the chart \( p: C^* \rightarrow \mathbb{C}^* \), but does not extend to a meromorphic 1-form on \( \mathbb{C}^* \).

It turns out it is often advantageous to consider \( C^* \) 1-forms instead of just holomorphic and meromorphic 1-forms. Locally, these are given by

\[
  w = f(x,y) \, dx + g(x,y) \, dy
\]

where \( x \) and \( y \) are the local real coordinates, i.e., \( z = x + iy \).
At is much more convenient to stick with working with $z$. Note

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}.$$  Thus, we have

$$dx = \frac{dz + d\bar{z}}{2}$$

and

$$dy = \frac{dz - d\bar{z}}{2i}.$$

Thus, instead of writing $f(x,y)dx + g(x,y)dy$, we can write

$$w = r(z, \bar{z})dz + s(z, \bar{z})d\bar{z}.$$  We will start to use $dz$ and $d\bar{z}$

instead of $dx$ and $dy$.

**Exercise:** Show one can rewrite partial derivatives with respect
to $x$ and $y$ in terms of $z$ and $\bar{z}$. Use this to show

a. A function $f$ is holomorphic on an open set $V \subseteq \mathbb{C}$

iff \[
\frac{\partial f}{\partial \bar{z}} = 0.
\]

**Def:** A \textbf{$C^\infty$ 1-form} on an open set $V \subseteq \mathbb{C}$ is an expression

$w$ of the form

$$w = f(z, \bar{z}) dz + g(z, \bar{z}) d\bar{z}$$

where $f, g$ are $C^\infty$ functions on $V$. We say $w$ is a

$C^\infty$ 1-form in the variable $z$. 


Def: Let $w = f(z, \bar{z}) dz + g(z, \bar{z}) d\bar{z}$ be a $C^\infty$ 1-form in the coordinate $z$ defined on an open set $V_1$, and $w = f(z, \bar{z}) dz + g(z, \bar{z}) d\bar{z}$ a $C^\infty$ 1-form defined on an open set $V_2$. Let $\omega = T(w)$ be a holomorphic mapping of $V_2$ to $V_1$. We say $w_1$ transforms to $w_2$ under $T$ if $f_2(w, \bar{w}) = f_1(T(w), \bar{T(w)}) T'(w)$ and $g_2(w, \bar{w}) = g_1(T(w), \bar{T(w)}) T'(w)$.

Note we move the $dz$ part to the $dw$ part and the $d\bar{z}$ to the $d\bar{w}$ part. There are no cross terms. This is a major reason to use $z$ and $\bar{z}$ instead of $x$ and $y$!

Def: Let $X$ be a Riemann surface. A $C^\infty$ 1-form on $X$ is a collection of $C^\infty$ 1-forms $\omega_\phi$, one for each chart $\phi: U \to V$, such that if two charts $\phi_i: U_i \to V_i$ have overlapping domains, the associated $C^\infty$ 1-form $\omega_\phi$ transforms to $\omega_{\phi_2}$ under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

As with hol. and mero. forms, one only needs to define $C^\infty$ 1-forms on an atlas to be able to uniquely extend to all charts.
When we integrate along contours we will make use of 1-forms to define this. We will also want to define surface integrals. To do this, we need 2-forms.

**Def:** A \( C^\infty \) 2-form on an open set \( V \subseteq \mathbb{C} \) is an expression \( \eta \) of the form

\[
\eta = f(z, \bar{z}) \, dz \wedge d\bar{z}
\]

where \( f \) is a \( C^\infty \) function on \( V \).

If you are familiar with exterior algebras you have seen wedge products before. If not, we define formally that

\[
dz \wedge d\bar{z} = 0 \neq d\bar{z} \wedge dz
\]

and

\[
dz \wedge d\bar{z} = -d\bar{z} \wedge dz.
\]

Note changing the order here amounts to reversing the orientation of the surface. The fact that \( dz \wedge d\bar{z} = 0 \neq d\bar{z} \wedge dz \) corresponds to the fact that one can't have a surface integral in only one variable.

**Def:** Let \( \eta_1 = f(z, \bar{z}) \, dz \wedge d\bar{z} \) be a 2-form on \( V_1 \) and \( \eta_2 = f(z, \bar{z}) \, dw \wedge d\bar{w} \) a 2-form on \( V_2 \). Let \( T : \mathbb{C} \to \mathbb{C} \) define a holomorphic map from \( V_2 \) to \( V_1 \). We say \( \eta_1 \) transforms to \( \eta_2 \) under \( T \).
If 

\[ f(z, \bar{z}) = f(T(z), \overline{T(z)}) \text{ for } T(z) \text{ close to } z. \]

This comes from requiring 

\[ dz \wedge d\bar{z} = T'(z) \overline{T'(z)} \, dz \wedge d\overline{z} \]

\[ = T'(z) \overline{T'(z)} \, dw \wedge d\overline{w} \]

\[ = || T'(w) ||^2 \, dw \wedge d\overline{w}. \]

The definition of a $C^\infty$ 2-form on a Riemann surface is exactly as one would expect from the earlier definitions.

**Def:** Let $X$ be a Riemann surface. A $C^\infty$ 2-form on $X$ is a collection of $C^\infty$ 2-forms $\Omega_{\phi_i}$, one for each chart $\phi_i : U_i \rightarrow \mathbb{C}$, such that on two charts $\phi_i : U_i \rightarrow \mathbb{C}$, $\Omega_{\phi_i}$ have overlapping domains then the associated $C^\infty$ 2-form $\Omega_{\phi_i}$ transforms to $\Omega_{\phi_j}$ under the change of coordinate mapping $T = \phi_i \circ \phi_j^{-1}$.

Once again, it is enough to specify $C^\infty$ 2-forms on an atlas to obtain a $C^\infty$ 2-form on all charts of $X$.

We can perform various operations on our forms and obtain new forms. Given a $C^\infty$ function $h$ and a $C^\infty$ 1-form $\omega$, we have $h \omega$ is a $C^\infty$ 1-form obtained by locally writing

\[ h \omega = (h \circ T') \omega \]

\[ = (h(\phi_i(z))) \omega_{\phi_i}(z) \]

\[ = \phi_i^* h \omega. \]
\( w = f dz + g dz \) and setting \( h w = h f dz + h f dz \). Moreover, we have the following operations:

1. If \( w \) is of type \((1, 0)\), so is \( h w \).
2. If \( w \) is of type \((0, 1)\), so is \( h w \).
3. If \( w \) and \( h \) are holomorphic, so is \( h w \).
4. If \( w \) and \( h \) are meromorphic, so is \( h w \). Moreover, we have \( \text{ord}(h w) = \text{ord}(h) + \text{ord}(w) \).
5. If \( h \) is \( C^\infty \) and \( \eta \) is a \( C^\infty \) 2-form, then \( h \eta \) is a \( C^\infty \) 2-form given locally by \( h f dz \wedge dz \) \( \eta \) is given locally by \( f dz \wedge dz \).

From what we have so far, we can now define several sheaves:

- \( L^1_X \) = sheaf of holomorphic 1-forms
- \( H^1_X \) = sheaf of meromorphic 1-forms
- \( \mathcal{E}^\infty_X \) = sheaf of \( C^\infty \) functions
- \( \mathcal{E}^{11}_X \) = sheaf of \( C^\infty \) 1-forms.

Note we have the following containment of sheaves:

\( \mathcal{O}_X \subseteq \mathcal{E}^\infty_X \)
and \( \Omega_x^1 \leq M_x^{(1)} \). Moreover, we also have that

\( \Omega_x^1 \) and \( M_x^{(1)} \) are sheaves of \( C \)-algebras and both are modules over the sheaf \( \mathcal{O}_x \). Similarly, \( \mathcal{E}_x^{(1)} \) is a sheaf of \( C \)-algebras and a module over \( \mathcal{E}_x^\infty \).

We saw before that the "holomorphic" and "anti-holomorphic" parts of a \( C^\infty \) 1-form do not mix under a transformation. This allows us to define forms that only involve \( dz \) or \( d\bar{z} \).

**Def:** A form \( \mathcal{E}_x^{(1)}(X) \) is of type \( (1,0) \) if it is locally of the form \( f(z, \bar{z}) \, dz \). It is of type \( (0,1) \) if it is locally of the form \( g(z, \bar{z}) \, d\bar{z} \).

Given \( U \subseteq X \) an open set, let \( \mathcal{E}_x^{(1,0)}(U) \) denote the set of \( C^\infty \) 1-forms of type \( (1,0) \) and similarly for \( \mathcal{E}_x^{(0,1)}(U) \).

**Exercise:** 1) Show \( \mathcal{E}_x^{(1,0)} \) and \( \mathcal{E}_x^{(0,1)} \) form sheaves of \( C \)-algebras.

2) Show \( \Omega_x^1 \) is a subsheaf of \( \mathcal{E}_x^{(1,0)} \).

3) Show the definition for \((1,0)\) forms is well-defined, namely, if \( \omega \) is a \((1,0)\) form in one chart, it is a
(1,0) form in all charts.

7) For any \( U \subseteq X \) open, show \( \xi_x^{(1)}(U) = \xi_x^{(1,0)}(U) \oplus \xi_x^{(0,1)}(U) \).

There are plenty of maps between these sheaves as well. Let

\( X \) be a Riemann surface and \( f \in \mathcal{C}^\infty(X) \). Let \( \phi: U \to V \) be

a chart on \( X \) in the local coordinate \( z \). Write \( f \) in these coordinates

as \( f(z, \bar{z}) \). Then we define:

\[
\partial f = \frac{\partial f}{\partial z} \, dz
\]

\[
\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} \, d\bar{z}
\]

and

\[
df = \partial f + \bar{\partial} f.
\]

**Lemma 3.2:** The local definitions \( \partial f, \bar{\partial} f, \text{ and } df \) give

rise to forms defined globally on \( X \). A \( C^\infty \) function

\( f \) is holomorphic iff \( \bar{\partial} f = 0 \). The operators \( d, \partial, \bar{\partial} \),

and \( \bar{\partial} \) are \( C \) linear and satisfy

\[
d(fg) = f \, dg + g \, df
\]

\[
\partial(fg) = f \, \partial g + g \, \partial f
\]

\[
\bar{\partial}(fg) = f \, \bar{\partial} g + g \, \bar{\partial} f.
\]
Proof: Exercise.

This gives rise to the following sheaf maps:

\[ d : \mathcal{E}_x^\infty \to \mathcal{E}_x^1 \]
\[ \partial : \mathcal{E}_x^\infty \to \mathcal{E}_x^{(1,0)} \]
\[ \bar{\partial} : \mathcal{E}_x^\infty \to \mathcal{E}_x^{(0,1)} \]

Let \( \mathcal{E}_x^{(1)} \) denote the sheaf of \( \mathcal{C}^\infty \) \( 2 \)-forms. Let \( \omega_1, \omega_2 \in \mathcal{E}_x^{(1)}(x) \)

and choose local coordinates so \( \omega_1 = f_1 dz + g_1 d\bar{z} \) and \( \omega_2 = f_2 dz + g_2 d\bar{z} \).

We have a \( \mathcal{C}^\infty \) \( 2 \)-form given by

\[ \omega_1 \wedge \omega_2 = (f_1 g_2 - f_2 g_1) \, dz \wedge d\bar{z} . \]

Exercise: Show \( \omega_1 \wedge \omega_2 \) is a well-defined form in \( \mathcal{E}_x^{(2)}(x) \), namely, show the local definition gives rise to a global form. Use the definitions and work out all the details.

We can also differentiate \( \mathcal{C}^\infty \) \( 1 \)-forms. Let \( \psi \in \mathcal{E}_x^{(1)}(x) \).

Let \( \phi : U \to V \) be a chart with local coordinates \( z \). Write

\[ \omega = f(z, \bar{z}) \, dz + g(z, \bar{z}) \, d\bar{z} . \]

Define

\[ \partial \omega = \frac{\partial f}{\partial \bar{z}} \, dz \wedge d\bar{z} . \]
\[ \partial w = \frac{\partial f}{\partial z} \, dz \wedge d\bar{z} = - \frac{\partial f}{\partial \bar{z}} \, dz \wedge d\bar{z}. \]

and
\[ dw = \partial w + \bar{\partial} w = \left( \frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) dz \wedge d\bar{z}. \]

Lemma 3.3: We have that \( \partial w, \partial \bar{w}, \) and \( dw \) are well-defined \( C^\infty \) 2-forms. We have \( \omega \in \Omega^2(X) \) iff \( \partial w = 0 \).

The operators \( \partial, \bar{\partial}, \) and \( d \) are all \( C^\infty \)-linear and satisfy
\[ d(fw) = df \wedge w + fdw, \]
\[ \partial(fw) = df \wedge w + f\partial w, \]
and
\[ \bar{\partial}(fw) = \bar{df} \wedge w + f \bar{\partial} w \]
for any \( f \in C^\infty(X) \). We also have
\[ dd f = \partial \bar{\partial} f = \bar{\partial} \partial f = 0 \]
for any \( f \in C^\infty(X) \).

Proof: Exercise.

**Def:** A function \( f \in C^\infty(U) \) is said to be harmonic if
\[ \partial \bar{\partial} f = 0. \] We say \( \omega \in \Omega^k(X) \) is closed if \( d\omega = 0 \), \( \partial \)-closed if \( \partial \omega = 0 \), and \( \bar{\partial} \)-closed if \( \bar{\partial} \omega = 0 \). A form \( \omega \) is said to be exact if \( \exists f \in \Omega^k(X) \) such that \( df = \omega \).
Exercise: 1) Let $F \in \Omega_x'^{(0)}(X)$. Then $\omega = 0$, i.e., every zeroth
form is closed.

2) If $\omega \in \Omega_x'(X)$, then $\omega = 0$.

3) If $\omega \in \Omega_x'^{(1)}(X)$ and is closed, then $\omega = \Omega_x'(X)$.

The last operation on differential forms we will want before we move
to integration is the notion of the pullback of a differential. A word of
warning here. The terms that appear are completely in regard to an
indeterminate map or going. This can be very confusing,
but keep in mind pullback means back and uses an upper,
star. We will end up with the weird looking expression

$$F^* : \Omega^*_X \to F^* \Omega^*_X,$$

but just keep in mind what each is doing and it will make sense.

Let $X$ and $Y$ be Riemann surfaces and let $F : X \to Y$
be a holomorphic map, nonconstant. Recall that for a sheaf

$\mathcal{F}$ on $X$, we obtain a sheaf on $Y$, denoted $F^* \mathcal{F}$ defined

by: given $V \subseteq Y$ open, we set $F^* \mathcal{F}(V) = \mathcal{F}(F^{-1}(V))$.

In our situation, we consider the case where $F = \Omega_x'^{(1)}, \Omega_x'^{(0)},
\Omega_x'^{(1)}, \Omega_x'^{(0)}, \Omega_x'^{(0)}, \Omega_x'^{(0)}, \Omega_x'^{(0)}$. We pick the case $F = \Omega_x'^{(1)}$ for example.
We have a map of sheaves

\[ F^* : \mathcal{E}^{(0)}_Y \to F^* \mathcal{E}^{(1)}_X, \]

i.e., for each \( V \in \mathcal{Y} \) open, we have a map

\[ F^* : \mathcal{E}^{(1)}_Y(V) \to \mathcal{E}^{(1)}_X(F^{-1}(V)). \]

To see this, let \( \omega \in \mathcal{E}^{(1)}_Y(V) \). Locally this is defined by

\[ \omega = f(z, \bar{z}) \, dz + g(z, \bar{z}) \, d\bar{z} \]

for some \( f, g \) \( C^\infty \) functions. Locally we can write \( F \) in the form \( Z = h(u) \) for some holomorphic map \( h \). Then locally we have

\[ F^* \omega = f(h(u), \overline{h(u)}) \, h'(u) \, du + g(h(u), \overline{h(u)}) \, \overline{h'(u)} \, du. \]

**Exercise:** 1) Then \( F^* \omega \) is actually in \( \mathcal{E}^{(1)}_X(F^{-1}(V)) \).

2) Show that if
   
   a) \( \omega \in \mathcal{E}^{(1)}_Y(V) \), then \( F^* \omega \in F^* \mathcal{E}^{(1)}_X (V) \)
   
   b) \( \omega \in \mathcal{M}^{(1)}_Y(V) \), then \( F^* \omega \in F^* \mathcal{M}^{(1)}_X (V) \)
   
   c) \( \omega \in \mathcal{E}^{(1)}_Y(V) \), then \( F^* \omega \in F^* \mathcal{E}^{(1)}_X (V) \)
   
   d) \( \omega \in \mathcal{E}^{(1)}_Y(V) \), then \( F^* \omega \in F^* \mathcal{E}^{(1)}_X (V) \).

One can define the pullback of a differential 2-form analogously. One should work out the details as an exercise.
Prop. 3.4: Let $f$ be a $C^\infty$ function and $\omega$ a $C^\infty$ 1-form.

Then

1) $F^*(df) = dF^*(f)$ and $F^*(d\omega) = dF^*(\omega)$

where $F^*(f) = f \circ F$.

2) $F^*(df) = 2F^*(f)$ and $F^*(d\omega) = 2F^*(\omega)$

3) $F^*(\overline{df}) = \overline{2F^*(f)}$ and $F^*(\overline{d\omega}) = \overline{2F^*(\omega)}$.

Proof: Exercise.

Finally, we note that we can calculate the order at a point $p$ of the pullback of a meromorphic 1-form in terms of the multiplicity of the form and the map $F$.

Lemma 3.5: Let $F: X \to Y$ be a meromorphic holomorphic map between Riemann surfaces and $\omega \in \Omega_Y^1(Y)$. Fix a point $p \in X$. Then

$$\text{ord}_p (F^*\omega) = (1 + \text{ord}_{F(p)}(\omega)) \text{mult}(F) - 2.$$ 

Proof: Pick local coordinates $w$ near $p$ and $z$ near $F(p)$ so that $F$ has local normal form $z = w^n$ for $n = \text{mult}(F)$.

With respect to these coordinates, if $k = \text{mult}_{F(p)}(\omega)$, then

$$w = (cz^k + \text{higher degree terms})dz.$$ Thus, we have
\[ F^*(w) = (C W^{nk} + \text{higher degree term}) (n W^{n-1}) dW. \]

Thus,

\[ \text{ord}_p F^*(w) = nk + n - 1, \]

as desired.

We now move to integration. We have defined the forms, so we now define the continuons.

**Def:** A path on a Riemann surface \( X \) is a continuous and piecewise \( C_0 \)

function \( Y : [a,b] \to X \) from a closed interval in \( \mathbb{R} \) to \( X \). The

points \( Y(a) \) and \( Y(b) \) are the endpoints of the path. We say

the path is closed if \( Y(a) = Y(b) \).

**Example:** Let \( Y : [a,b] \to X \) be a path. Then \(-Y\) is the reversed

path defined by \(-Y(t) = Y(a + b - t)\).

**Example:** Let \( F : X \to Y \) be a \( C^0 \) map. If \( Y \) is a path

on \( X \), then \( F \circ Y \) is a path on \( Y \).

**Example:** Let \( p \in X \) and let \( S \subseteq X \) so that \( p \notin S \). This is a closed

path \( Y \) on \( X \) so that:
1) $f$ is 1-1 and the image of $f$ lies completely inside the domain $U$ of a chart $\phi: U \to V$.

2) The closed path $\phi \circ \gamma$ on $V$ has winding number $1$ about the point $\phi(p)$.

3) No point of $S$ which lies in the domain $U$ is mapped to the interior of $\phi \circ \gamma$, i.e., for every $s \in S \cap U$, the winding number of $\phi \circ \gamma$ about $\phi(s)$ is zero.

We say such a path is a small path enclosing $p$ but enclosing any point of $S$.

Note: we can arrange by our choice of chart that

1) $\phi$ is centered at $p$
2) the domain of $Y$ is $10, 2\pi$.
3) the closed path $\phi \circ \gamma$ is exactly the path

$z(t) = r \exp(it)$ for some real number $r > 0$.

**Lemma 3.6**: Let $Y$ be a path on a Riemann surface $X$. Then $X$ may be partitioned into a finite number of paths $\gamma_i$, such that each $Y_i$ is $C^\infty$, with image contained in a single chart domain of $X$.

**Proof**: Exercise.

We can now integrate by switching to local coordinates.
Let $w = \mathcal{E}_x^{(1)}(x)$ and let $Y$ be a path in $\mathcal{X}$. Choose a partition of the domain of $\mathcal{S}$, say $[a_i, a_{i+1}]$, so that each $\Delta_i = \mathcal{S}[a_i, a_{i+1}]$ is contained in the domain $\mathcal{U}_i$ of a chart $\phi_i$. With respect to the chart $\phi_i$, write $w = f_i(z, \overline{z}) \, dz + g_i(z, \overline{z}) \, d\overline{z}$. (Note really it should be $z_i, \overline{z}_i$ as well!) The composition $\phi_i \circ \mathcal{S}_i$ defines a function $z_i(t)$ for $t \in [a_i, a_{i+1}]$. We then define the integral of $w$ along $Y$ by

$$\int_Y w = \sum_i \int_{a_i}^{a_{i+1}} \left( f_i(z(t), \overline{z(t)}) \, z'(t) \, dt + g_i(z(t), \overline{z(t)}) \, \overline{z'(t)} \, dt \right).$$

If we are in the fortunate situation that $Y$ lies entirely in one chart $\phi : U \to \mathbb{C}$, then we write $w = f \, dz + g \, d\overline{z}$ in this chart and have

$$\int_Y w = \int_{\phi \circ Y} f \, dz + g \, d\overline{z}.$$

**Exercise:** Show that because of how we chose transforms of 1-forms with respect to charts, this definition does not depend on the choice of charts.

The following result gives some of the basics of integration in this setting.
\textit{Lemma 3.7:} a) \textit{If \( \alpha \) is a reparametrization of the domain of \( Y \),}
\textit{i.e., \( \alpha : [c, d] \rightarrow (a, b) \) is a continuous \( C^\infty \) function sending \( c \) to \( a \) and \( d \) to \( b \), then}
\[ \int_Y \omega = \int_Y \omega. \]

b) \textit{The integral is \( C^0 \)-linear in \( \omega \):}
\[ \int_Y (\lambda_1 \omega_1 + \lambda_2 \omega_2) = \lambda_1 \int_Y \omega_1 + \lambda_2 \int_Y \omega_2. \]

c) \textit{The fundamental theorem of calculus holds: if \( f \in C^0_x(U) \)}
\textit{where \( U \) is a nbh of the image of \( \gamma : [a, b] \rightarrow X \), then}
\[ \int_Y df = f(y(b)) - f(y(a)). \]

d) \textit{If \( Y \) is partitioned into paths \( \gamma_i \), then}
\[ \int_Y \omega = \sum_i \int_{\gamma_i} \omega. \]

e) \textit{We have}
\[ \int_Y \omega = -\int_Y \omega. \]

f) \textit{If \( F : X \rightarrow Y \) is a holomorphic map between R.S., then for}
\textit{\( Y \) a path on \( X \) and \( \omega \) a 1-form on \( Y \), we have}
\[ \int_{F \gamma} \omega = \int_{\gamma} F^* \omega. \]
\textit{We usually write \( F^* \) because it is pushing \( Y \) to \( Y \).}
Def: A chain on a Riemann surface $X$ is a formal finite sum of paths with integer coefficients.

The set of chains forms a free abelian group $CH(X)$. The basis is the set of paths. Let $Y = \sum_j n_j Y_j$ be a chain, and let $\xi^m_x(Y)$. We define the integral of $w$ over $Y$ by

$$\int_Y w = \sum_j n_j \int_{Y_j} w.$$ 

This gives integration is a bilinear operation: $C^1$ - linear in the 1-forms and $\mathbb{Z}$ - linear in the chain.

We end this chapter by looking at residues of meromorphic 1-forms and some familiar results that follow from studying these. Let $\omega$ be a 1-form on $X$ that is meromorphic at $p \in X$. We choose local coordinate $z$ centered at $p$ and write

$$\omega = \frac{f(z)}{z^{p+1}}dz$$

$$= \left( \sum_{n=-M}^{\infty} c_n z^n \right)dz$$

where $c_{-M} \neq 0$, i.e., $\mathrm{ord}_p(\omega) = -M$. We make the following definition.
Def: The residue of \( w \) at \( p \), denoted \( \text{Res}(w) \), is the coefficient \( c_1 \) in the above expansion.

Note we have seen before that Laurent series are not well-defined. It turns out the residue is well-defined.

Lemma 3.8: Let \( w \) be a meromorphic 1-form defined in a nbhd of \( p \in X \). Let \( Y \) be a small path on \( X \) enclosing \( p \) and not enclosing any other pole of \( w \). Then

\[
\text{Res}_p(w) = \frac{1}{2\pi i} \int_Y w.
\]

Proof: Let \( \phi: U \rightarrow V \) be a chart on \( X \) centered at \( p \) containing the image of \( Y \) so that \( Y \) is a small path enclosing \( p \) wrt the chart. Write \( w = \sum_{n=1} \frac{c_n}{z^n} \) in \( V \) and let \( \phi(z) = \sum_{n=1} c_n z^n \) be the Laurent series. Then

\[
\int_Y w = \int_{\phi(Y)} \frac{\phi'(z)}{z} dz.
\]

Now we use the residue theorem from complex analysis to get

\[
\int_{\phi(Y)} \frac{\phi'(z)}{z} dz = 2\pi i c_1.
\]

This gives the result.

In particular, we see the residue is well-defined.
Our next step is to integrate 2-forms. Now we need surface instead of curves. We say \( T \) is a triangle in \( X \) if it is the homeomorphic image of a triangle in \( C \). Suppose \( T \) is completely contained in the domain of a chart \( \phi: U \to V \). Let \( \eta \) be a \( C^\infty \) 2-form on \( X \) and with \( \eta = f(z, \overline{z}) dz \wedge \overline{dz} \) with respect to this chart. We define

\[
\iint_T \eta = \iint_{\phi(T)} f(z, \overline{z}) dz \wedge \overline{dz}
\]

\[
= \iint_{\phi(T)} (-2i) f(x+iy, x-iy) \, dx \wedge dy.
\]

**Exercise:** Show this integral is well-defined, namely, it does not depend on the choice of chart.

In general our triangles will not fit in the domain of a single chart. Let \( D \subseteq X \) be a triangulizable closed set. Then to integrate over \( D \), we add up the integrals over the triangles that we make small enough to fit in charts. You should show this to be well-defined.

**Given any triangle** \( T \subseteq X \), we can form a closed path \( \partial T \) consisting of the boundary of \( T \) counterclockwise parameterized.
by arc lengths. For \( D \) triangulable, we write \( \partial D \) for the
sum of the boundaries of the triangle triangulating \( D \). We can
now give Stokes' Theorem.

**Theorem 3.9 (Stokes' Theorem):** Let \( D \) be a triangulable closed set in
a Riemann surface \( X \). Let \( w \in \mathcal{E}^{(1)}(U) \), for \( U \) an open,
set containing \( D \). Then

\[
\oint_{\partial D} w = \iint_D dw.
\]

**Proof:** One triangulate \( D \) with small triangles and then transfer
the integrals to the plane. Now it is Green's Area Form calculus.

The last result of this chapter is the Residue Theorem for compact
Riemann surfaces. Recall the residue theorem from complex analysis.
It states that if \( f \) is a meromorphic function on an open set \( U \)
and \( \gamma \) is a positively oriented simple closed curve that contains
all the poles of \( f \), then

\[
\oint_{\gamma} dz = 2\pi i \sum \text{Res}(f).
\]

where the \( z_k \) are the poles of \( f \). The statement for a compact Riemann
Theorem 3.10 (The Residue Theorem): Let \( \omega \in \mathcal{M}_X^{(1)}(X) \) with \( X \) a compact Riemann surface. Then

\[
\sum_{p \in X} \text{Res}_p(\omega) = 0.
\]

Proof: Note that we have seen the poles of \( \omega \) give a discrete set in \( X \). Since \( X \) is compact, this gives us a finite set of poles and so the sum is a finite sum. Let \( p_1, \ldots, p_n \) be the poles of \( \omega \). For each \( i \), choose a small path \( \gamma_i \) in \( X \) that encloses \( p_i \) but not \( p_j \) for \( j \neq i \). Let \( U_i \) be the interior of \( \gamma_i \). We have via Lemma 3.8 that

\[
2 \pi i \text{ Res}_p(\omega) = \int_{\gamma_i} \omega.
\]

Let \( D = X - \bigcup U_i \). We have \( D \) is a compact Riemann surface \( \Delta \subseteq D \) is triangulable, and \( 2D = -\sum \gamma_i \) is a chain in \( X \). Thus,

\[
\sum_{i} \text{ Res}_{p_i}(\omega) = \frac{1}{2\pi i} \sum_{i} \int_{\gamma_i} \omega
\]

\[
= \frac{1}{2\pi i} \int_{2D} \omega
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \omega
\]

\[
= \frac{1}{2\pi i} \int_{\partial D} \omega
\]

(99)
\[ = -\frac{1}{2\pi i} \oint_D dw \quad (\text{By Stokes'}) \]

\[ = 0 \]

where we have used \( dw = 0 \) \( \forall \) \( w \) \( \text{in holomorphic in a neighborhood of} \ D. \)

One can use this result to give an analytic proof of the Riemann-Roch theorem. We will instead focus on an algebraic proof, but it is worth noting the power of the residue theorem. We also get as an immediate corollary the following result we have already shown.

**Corollary 3.11:** Let \( f \) be a nonconstant meromorphic function on a compact Riemann surface \( X \). Then

\[ \sum_{p \in X} \text{ord}_p(f) = \chi. \]

**Proof:** Exercise. (Hint: Set \( u = \frac{df}{f} \) and observe

\[ \text{Res}_p \left( \frac{df}{f} \right) = \text{ord}_p(f). \]