Chapter 5  Algebraic Curves and Riemann-Roch:

Our first step in this chapter is to introduce algebraic curves. Recall that in proving that a canonical divisor on a compact Riemann surface of genus $g$ has deg $2g-2$, we needed to assume $X$ had a global meromorphic function. From the point of view of this course, an algebraic curve will be a compact Riemann surface with enough meromorphic functions. We now make this more precise.

Throughout this chapter we take $X$ to be a compact Riemann surface of genus $g$.

Let $S$ be a set of meromorphic functions on $X$. Let $p, q \in X$ be distinct points. We say $S$ separates the points $p$ and $q$ if there exist $f \in S$ so that $f(p) \neq f(q)$. In particular, if $f$ has a pole at one and root at the other, this constitutes a separation.

We will also want the notion of separation of tangents. Here we restrict to manifolds for a moment to make the
ideas more clear. Let \( Y, Z \) be manifolds and \( F \) a map \( F : X \to Y \). We naturally have that the derivative map at a point \( p \in X \), \( D_p F \) is a linear map from the tangent space \( T_p(X) \) to the tangent space \( T_{F(p)}(Y) \):

\[
T_p(X) \xrightarrow{D_p F} T_{F(p)}(Y)
\]

In this setting, we say \( F \) separates tangents at \( p \) if \( D_p F \) is 1-1.

We translate this notion to our setting. Let \( f : U_X \to \mathbb{C} \) be defined at \( p \in X \) and consider the associated map \( F : X \to \mathbb{C} \).

Since \( f \) is defined at \( p \), we can choose coordinate centered at \( p \) so \( F \) can be written as a holomorphic function in these coordinates taking \( 0 \) to \( 0 \). The tangent space in \( \mathbb{C} \) is 1-dimensional, so the derivative map at \( 0 \) is just multiplication by \( g'(0) \).

Thus, in this set-up we say \( f \) separates tangents at \( p \) if 
\[
g'(0) \neq 0.
\]
Exercise: 1) Let $p \in X$. Show $F$ has $\text{mult}(F) = 1$ iff the following is satisfied: either $f$ is holomorphic at $p$ and $\text{ord}_p(f - f(p)) = 1$ or $f$ has a simple pole at $p$.

2) Show $F$ separates tangents at $p$ iff $\text{mult}(F) = 1$.

Definition: A compact Riemann surface is an algebraic curve if the field $M_x(X)$ separates tangents and points of $X$.

Exercise: Show the following are algebraic curves:

1) $C_i$ (use rational functions)

2) $C/L$ (use ratio of theta functions, see lecture or substitute next week)

3) Any smooth projective plane curve (use ratios of homogeneous polynomials of the same degree.)

One important thing to note is $y \in X$ is an algebraic curve, then for every $p \in X$ we can find $f \in M_x(X)$ s.t.

$\text{ord}_p(f) = 1$. Take $g$ separating tangents at $p$ and use either $f = g - g(p)$ if $g$ is holomorphic at $p$ or $f = \frac{1}{g}$ if $g$ has a simple pole at $p$. 
This brings us to the following deep result. This is rightfully an analysis theorem so we will not attempt any proof here.

**Theorem 5.1:** Every compact Riemann surface is an algebraic curve.

With this in mind, we will often refer to $M_x(X)$ as the function field of $X$ and write it as $M(X)$.

**Prop. 5.2:** Let $X$ be an algebraic curve. Then the function field $M(X)$ is a field extension of $\mathbb{C}$ of transcendence degree 1.

**Proof:** Since $X$ is algebraic, there is a nonconstant meromorphic function on the transcendence degree is at least 1.

Suppose $f$ and $g$ are algebraically independent elements of $M(X)$. Let $D$ be a divisor s.t. $D \geq \text{div}_x(f)$ and $D \geq \text{div}_x(g)$. Then $f, g \in L(D)$. Now, for any $i, j \geq 0$, we have $f^i g^j \in L(nD)$.
if \( i \neq j \leq n \). Thus, \( L(n^2) \) contains every monomial of the form \( x_i x_j \) for \( i \neq j \leq n \). Note that one has \( \frac{n^2 + 3n + 2}{2} \) such monomials (exercise!) Moreover, since \( f \) and \( g \) are algebraically independent, all of these monomials are linearly independent. Thus

\[
\dim \ L(n^2) \leq \frac{n^2 + 3n + 2}{2}.
\]

On the other hand, we have

\[
\dim \ L(n^2) \leq \deg(n^2) + 1 = 1 + n \deg(D).
\]

This is a contradiction for large \( n \). Thus, \( f \) and \( g \) cannot be algebraically independent. \( \square \)

We also have the following result. The proof can be found in the textbook.

Prop. 5.3: The field \( \mathbb{C}(x_1, \ldots, x_n) \) is finitely generated over \( \mathbb{C} \).

Though we are still a bit of work away from proving it, we can now state the Riemann-Roch Theorem and see
some elementary applications. We will come back to the proof later.

**Theorem 5.4 (Riemann–Roch):** Let \( X \) be an algebraic curve of genus \( g \). Then for any divisor \( D \) and any canonical divisor \( K \) we have

\[
\dim_\mathbb{C} \mathcal{L}(D) - \dim_\mathbb{C} \mathcal{L}(K-D) = \deg D + 1 - g.
\]

To save writing, we will denote \( \dim_\mathbb{C} \mathcal{L}(D) \) by \( \ell(D) \) as is customary.

For "big" divisors, this immediately allows us to calculate \( \ell(D) \).

**Corollary 5.5:** Let \( D = D + \text{div}(f) \) with \( \deg D > 2g - 1 \). Then

\[
\ell(D) = \deg D + 1 - g.
\]

**Proof:** Recall \( \deg K = 2g - 2 \). Thus, \( \deg (K-D) = \deg K - \deg D < 0 \).

This gives \( \dim_\mathbb{C} \mathcal{L}(K-D) = 0 \).

We will now give several applications of R.R. before coming back to the proof; next chapter. The first result shows that if a compact Riemann surface satisfies the R.R. theorem,
Prop. 5.6: Let $X$ be a compact R.R. that satisfies the R.R. Theorem for every divisor $D$. Then $X$ is an algebraic curve.

Proof: We must show $M(X)$ separate points and tangent to $X$. Let $p, q \in X$, $p \neq q$. Let $D = (g+1)p$. Since $\ell(X-D) \geq 0$, we have $\ell(D) = \deg(D) + 1 - g = 2$. Thus, $f$ is a nonconstant $f \in \mathcal{L}(D)$. We have $f$ has no pole at $p$, but no other poles as $f$ separate $p$ and $q$. Thus, $M(X)$ separate points.

For $p \in X$ and consider divisors $D_n = n - p$. We know for large $n$ that $\ell(X-D_n) = 0$, so

$$\ell(D_n) = \deg(D_n) + 1 - g = n + 1 - g.$$ 

Thus, for large $n$ there are functions in $\mathcal{L}(D_n) - \mathcal{L}(D_n)$. Thus, for large $n$ there are functions $f$ that have pole of exact order $n_0$ at $p$ and no other poles. Then $f$ has a simple zero at $p_0$, and so separate tangents at $p$.

We can use R.R. to classify curves of small genus. We begin with genus 0 curves as they require no more background.
Lemma 5.7: Let $X$ be a compact R.S. Suppose for some $p \in X$, $L(p) \not= 0$. Then $X$ is isomorphic to $C_\omega$.

Proof: Assume $L(p) \not= 0$, then $L$ is a meromorphic function on $X$. This function must have poles because it is meromorphic on a compact R.S. However, the only pole allowed is a simple one at $p$. Thus, the associated map $F: X \to C_\omega$ is degree 1, so an isomorphism.

Prop 5.8: Let $X$ be an algebraic curve of genus 0. Then $X \cong C_\omega$.

Proof: Fix $p \in X$. We know $\deg(K) = 2g - 2 = -2$, we have $\deg(K - p) = -3$, so $L(K - p) = 0$. Thus, $L(p) = \deg(p) + g = 2$. We now apply Lemma 5.7 to get the result.

Cor 5.9: Let $X$ be an algebraic curve that is not isomorphic to $C_\omega$. Then, for any $p \in X$ we have $L(p) \not= 0$. In particular, this follows for any $X$ of genus not 0.

Proof: Exercice.
We will make use of the following result when classifying algebraic curves.

**Prop. 5.10:** Every algebraic curve can be holomorphically embedded into projective space.

The proof is very easy using R.R., but we need more background on some special types of divisors and maps to projective space first.

**Def.:** Let \( X \) be a R.S. A map \( \phi: X \to \mathbb{P}^n \) is **holomorphic** at \( p \in X \) if there are holomorphic functions \( g_0, \ldots, g_n \) defined on \( X \) near \( p \), not all zero at \( p \), so that
\[
\phi(x) = [g_0(x) : \ldots : g_n(x)] \quad \text{for} \quad x \text{ near } p.
\]
We say \( \phi \) is a **holomorphic map** if it is holomorphic at each point \( p \in X \).

**Exercise:** Show this agrees with our earlier definition of holomorphic maps between R.S. when \( n = 1 \).

Since we are interested in compact R.S., the above definition is not very useful if we want to use the same \( g_i \) at each \( p \)
because this would force them to all be constant. We usually work with meromorphic functions instead. Let \( f = (f_0, \ldots, f_n) \) where the \( f_i \in \mathcal{M}(X) \) and are not all 0. Define \( \Phi_f : X \to \mathbb{P}^n \) by setting

\[
\Phi_f(p) = [f_0(p) : \ldots : f_n(p)].
\]

We have \( \Phi_f \) is defined at \( p \) if

- \( p \) is not a pole of any \( f_i \)
- \( p \) is not a common zero of the \( f_i \).

Whenever it is defined \( \Phi_f \) is a holomorphic map. In fact, we can extend the definition of \( \Phi_f \) to all of \( X \).

**Lemma 5.11**: All the functions \( f_1, \ldots, f_n \) are not all identically 0, then \( \Phi_f : X \to \mathbb{P}^n \) extends to a holomorphic map on all of \( X \).

**Proof**: Fix \( p \in X \) and set \( m = \min \text{ord}_p(f_i) \). For \( m \neq 0 \):

- if \( m < 0 \) then \( p \) is a pole of some \( f_i \) and if \( m > 0 \) \( p \) is a zero of each \( f_i \).

On a small nbhd of \( p \) we can assume there are no poles of any of the \( f_i \) except at possibly \( p \). Let \( z \) be a local coordinate centered at \( p \). Then every \( f_i(z) \) is
holomorphic for \( z = 0 \), \( z \neq 0 \). There is also no \( z \) near 0 that is a common zero to all the \( f_i \). Thus, for \( z \neq 0 \) we can multiply \( f_i(z) \) by \( z^{-m} \) without changing the value of \( \phi \); since we are in projective space, let \( g_i(z) = z^{-m}f_i(z) \).

Then

\[
\phi(z) = \left[ g_0(z) : \ldots : g_n(z) \right], \quad z \neq 0
\]

\[
= \left[ z^{-m}g_0(z) : \ldots : z^{-m}g_n(z) \right], \quad z \neq 0
\]

\[
= \left[ g_0(z) : \ldots : g_n(z) \right].
\]

The last expression for \( \phi \) is holomorphic in each coordinate near 0 with at least one coordinate nonzero. Thus, \( \phi \) is well-defined at 0 by setting \( \phi(0) = [g_0(0) : \ldots : g_n(0)] \). This extends \( \phi \).

In fact, every holomorphic map arises in this manner.

Prop. 5.12: Let \( \phi : X \to \mathbb{P}^n \) be a holomorphic map. Then there are meromorphic functions \( f_0, \ldots, f_n + M(X) \) so that \( \phi = \phi_f \) with \( f = (f_0, \ldots, f_n) \). Moreover, if two tuples \( (f_0, \ldots, f_n) \) and \( (g_0, \ldots, g_n) \) satisfy \( \phi_f = \phi_g \),
no holomorphic maps to $\mathbb{P}^n$, then there exist $\lambda \in M(X)$

so that $g_i = \lambda f_i$ for each $i$.

Proof: Exercise or read in the textbook. 

**Defn:** Let $D \in \text{Div}(X)$. The complete linear system of $D$ is

$$\text{ldl} = \{ C \in \text{Div}(X) : E \sim D \text{ and } E \geq 0 \}$$

A linear system is a subset $Q$ of some $\text{ldl}$ so that

$$V = \{ f \in M(X) \mid f = 0 \text{ on } f \notin Q \}$$

is a $C$-vector space.

Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Write $\phi = (f_0, \ldots, f_n)$

where each $f_i \in M(X)$. We can associate a linear system to $\phi$ as follows. Let $D = -\text{min} \{ \text{div}(f_i) \}$. So for $p \in X$, we have

$$-D(p) = \text{min} \{ \text{ord}_p(f_i) \}.$$ 

In particular, for each $p$ and each $i$, we have $-D(p) \leq \text{ord}_p(f_i)$. This gives $f_i \in L(D)$ for each $i$. If we let $V_p$ be the $C$-span of $f_0, \ldots, f_n$.

Then $V_p \subseteq L(D)$ is a subspace. This gives that

$$\text{ldl} = \{ \text{div}(g) + D : g \in V_p \}$$

is a linear system which is a subsystem of $\text{ldl}$.

**Lemma 5.13:** The linear system $\text{ldl}$ is well-defined, i.e., it does
Proof: Suppose $\mathcal{I}$ is also given by $\mathcal{I}_{0,-\cdots,\mathcal{I}_{0}}$. Then the previous result gives $\lambda \in \mathcal{I}(x)$ so that $\mathcal{I}_{i} = \lambda f_i$. Thus,

$$\text{div}(\mathcal{I}_{i}) = \text{div}(\lambda) + \text{div}(f_i).$$

As $\lambda \in \mathcal{I}_{i}$ is associated as above to the $f_i$ and $\mathcal{I}_{i}'$ to the $g_i$, then $D = D - \text{div}(\lambda)$, and so $\mathcal{I} \subset \mathcal{I}'$. This gives $\mathcal{I}_{1} \subset \mathcal{I}_{1}'$.

Write $\Phi$ as the map coming from the $f_i$ and $g_i$ when using the $g_i$. We easily see $\lambda \Phi = \lambda \Phi_1 = 1_{\mathcal{I}_{1}}$ as follows. A typical element of $1_{\mathcal{I}_{1}}$ is a divisor of the form $\text{div}(\Sigma c_i g_i) + D_i$.

We have

$$\text{div}(\Sigma c_i g_i) + D_i = \text{div}(\Sigma c_i \lambda f_i) + D_i$$

$$= \text{div}(\Sigma c_i f_i) + \text{div}(\lambda) + D_i$$

$$= \text{div}(\Sigma c_i f_i) + D_i$$

and so $1_{\mathcal{I}_{1}} \subset 1_{\mathcal{I}_{1}}$. The reverse containment follows similarly.

We call the linear system $\mathcal{I}$ the linear system of the map $\Phi$.

We have the following basic property, all such linear systems have.
**Lemma 5.14:** Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Then for every $p \in X$ there is a divisor $E \leq 1$ so that $p \not\in \text{supp}(E)$, i.e., there is no point in $X$ supported by every divisor in $1\mathcal{O}_X$.

**Proof:** Let $p \in X$ and write $\phi = [f_0 : \ldots : f_n]$. Recall we have $D = -\min_j \text{div}(f_j)$. Suppose at $p$ we have $\min_j \text{ord}_p(f_j) = k$. Let $j$ be the index so that $\text{ord}_p(f_j) = k$.

Then $D(p) = -k$. We have $E = \text{div}(f_j) + D + 1\mathcal{O}_X$.

However, $E(p) = \text{ord}_p(f_j) + D(p) = 0$, so $p \not\in \text{supp}(E)$.

This property will essentially be enough to distinguish between linear systems of the form $1\mathcal{O}_X$ and those that do not arise from a holomorphic map.

**Def:** Let $\mathcal{Q}$ be a linear system on $X$. A point $p \in X$ is a base-point of $\mathcal{Q}$ if every divisor $E \in \mathcal{Q}$ contains $p$ in its support, i.e., $E \not\geq 1 \cdot p$. We say $\mathcal{Q}$ is base-point-free if it has no base points.
We have just seen that given \( \phi : X \to \mathbb{P}^n \) holomorphic, then

\[ \phi \] is trace-point-free.

We express the notion of trace-points in terms of functions.

Let \( Q \subseteq \text{ID}_1 \) be a linear system that sat inside the complete linear system \( \text{ID}_1 \). Let \( V \) be the corresponding matrix space. We know the divisors in \( Q \) are those of the form \( D + d \text{div}(f) \) for \( f \in V \). Now given \( f \in L(D) \) for every \( p \in X \) we have \( D(p) + \text{ord}_p(f) \geq 0 \). Thus, \( p \) is a trace point for \( Q \) iff for every \( f \in V \) we have

\[ D(p) + \text{ord}_p(f) \geq 1. \]

Assume \( f \in L(D) \) automatically if \( f \) is in \( V \), then we are paying \( f \in L(D-p) \). Summarising, we have:

\[ \text{Lemma 5.15: A point } p \in X \text{ is a trace point of the linear system } Q \subseteq \text{ID}_1 \text{ defined by } V \subseteq L(D) \text{ iff } V \subseteq L(D-p). \]

In particular, \( p \) is a trace point of the complete linear system \( \text{ID}_1 \) iff \( L(D-p) = L(D) \).

Another way to say this is that \( p \) is not a trace point of \( Q \) iff there is a function \( f \in V \) with \( \text{ord}_p(f) = -D(p) \) exists.

We can now use the previous lemma along with the bound

\[ H(0) \leq 1 \text{ord}_p(f) \]

given before that \( L(D) \leq 1 \text{ord}_p(f) \) when \( D - p \cdot N \), to conclude the following.
Prop. 5.16: Let $\mathcal{X}$ be a compact R.S. and $\mathcal{D} = \text{Div} \mathcal{X}$. Then $p \in \mathcal{X}$ is a base point of $1D$ if and only if $l(D-p) = l(D)$. Thus, $1D$ is base-point-free if and only if for every point $p \in \mathcal{X}$, $l(D-p) = l(D) - 1$.

Proof: Exercise.

It is possible to realize the linear system $1D$ in terms of hyperplane divisors. We will come back to this in a little bit. First we finish the basics.

Prop. 5.17: Let $Q \in 1D$ be a base-point-free linear system of projective dimension $n < \text{proj}(V) = \text{proj}(\mathcal{X})$, where $V$ has dimension $n$. Then there is a holomorphic map $\phi: \mathcal{X} \rightarrow \mathbb{P}^n$ such that $Q = \text{iz}$. Moreover, $\phi$ is unique up to the choice of coordinates in $\mathbb{P}^n$.

Proof: Let $V$ be the vector space corresponding to $Q$. This means the elements of $Q$ are of the form $D + \text{div}(f)$ for $f \in V$.

Let $f_0, \ldots, f_n$ be a basis of $V$. Then we have

$\phi = [f_0: \ldots : f_n]$ is the desired map. (Check as exercise.)

It remains to prove uniqueness. Suppose we have $\phi' = [g_0: \ldots : g_n]$ also has $Q = \text{iz}$.

Then the elements of $}$
1°) are of the form \(\text{div}(g) + D'\) when \(g\) is a linear comb.

of the \(g_i\) and \(D'\) is defined in terms of the \(g_i\). Since

\[ \text{div}(1) = \text{div}(1) \] we may change coordinates so we can assume

\[ \text{div}(f_i) + D = \text{div}(g_i) + D' \]. Set \( h_i = f_i / g_i \). Then \( \text{div}(h_i) = D' - D \)

is independent of \(i\). Thus, by adjusting the \(g_i\)'s by constant

factors we can assume there is a unique meromorphic function

\( h \) on \( X \) so that \( h = f_i / g_i \). Thus, \( \phi = \phi' \) and so \( \phi \) is

unique up to change of coordinates on \( P^n \).

The point of this result is it gives a 1:1 correspondence:

\[
\begin{cases}
\text{base-point-free} \\
\text{linear systems on X}
\end{cases}
\leftrightarrow
\begin{cases}
\text{holo. maps } \phi : X \to P^n \\
\text{linear degree} \to \text{leaves, div, changes}
\end{cases}
\]

(Monomial means the image is not contained in a hyperplane)

We want to use this, along with the power of the \text{Riemann}-\text{ Roch Theorem}

to construct holomorphic maps to projective space. The easiest way is

to use complete linear systems. The difficulty is that in general

they will not be base-point free. We can deal with this by

removing the base-point.
Let $\mathcal{D}$ be a complete linear system with base points. Let $F$ be the largest divisor that occurs in every divisor in $\mathcal{D}$, i.e., $F = \min \{ E : E \in \mathcal{D} \}$. Then we have by construction that $\mathcal{D} - F$ has no base points. Moreover, if $E \in \mathcal{D}$, then there exists $E' \in \mathcal{D} - F$ so that $E = F + E'$. We call $F$ the

**fixed divisor** of $\mathcal{D}$.

**Lemma 5.18**: If $F$ is the fixed divisor of $\mathcal{D}$, then

$$L(D - F) = L(D).$$

**Proof**: We have $F > 0$ since $F \in \mathcal{D}$, so $D - F \in \mathcal{D}$ which gives $L(D - F) \subseteq L(D)$. Let $f \in L(D)$. Then $\text{div}(f) + D > 0$. Thus, $\text{div}(f) + D \in \mathcal{D}$ and so we can write $\text{div}(f) + D = F + E'$ for some $E' \in \mathcal{D} - F$.

Thus, $\text{div}(f) + (D - F) = E' > 0$, so $f \in L(D - F)$.

This gives $L(D) \subseteq L(D - F)$, which completes the proof. 

The point is we can remove the base points without shrinking $L(D)$. Thus, it is enough to work with base point free $\mathcal{D}$ when constructing maps.
Given $D \in \text{Div}(X)$ with $\textbf{1D}$ base-point-free, we write $\phi_D$

for $\phi_{\textbf{1D}}$. Now that we know how to construct $\phi_D$, we really

want to know when $\phi_D$ is a holomorphic embedding. The 1st

step is to determine when $\phi_D$ is 1-1.

Lemma 5.19: Let $X$ be a compact R.S., and $D \in \text{Div}(X)$ with

$\textbf{1D}$ base-point-free. Then there is a basis $f_0, f_1, \ldots, f_n$

of $L(D)$ so that $\text{ord}_D(f_i) = -D(p)$ and $\text{ord}_D(f_i) > -D(p)$

for $i \geq 1$.

Proof: Assume $\textbf{1D}$ is base-point-free, $L(D-p) \leq L(D)$ is a
codimension one subspace. Let $f_1, \ldots, f_n$ be a basis

for $L(D-p)$. We extend this to a basis of $L(D)$

by adding a function $f_0 \in L(D) - L(D-p)$. Then

$\text{ord}_D(f_i) \geq -D(p) + 1 > -D(p)$

for $i \geq 1$. If $\text{ord}_D(f_0) > -D(p)$, then $f_0 \in L(D-p)$,

which is a contradiction. Thus, $\text{ord}_D(f_0) = -D(p)$.

Prop. 5.20: Let $X$ be a compact R.S., $D \in \text{Div}(X)$ with $\textbf{1D}$

base-point-free. Fix $p, q \in X$ with $p \neq q$. Then

$\phi_D(p) = \phi_D(q)$ iff $L(D-p-q) = L(D-p) = L(D-q)$. Thus,
$\phi$ is 1-1 iff for every pair of distinct points $p, q \in X$, we have $l(D-p-q) = l(D)-2$.

**Proof:** We know changing a basis for $L(D)$ amounts to a linear change of coordinates for $\phi$, we can check if $\phi_p(p) = \phi_q(q)$ using any basis for $L(D)$. We use the previous lemma to choose a basis. Using the basis we have

$$\phi_p(p) = [1:0:\ldots:0].$$

Thus, $\phi_p(p) = \phi_q(q)$ iff $\phi_q(q) = [1:0:\ldots:0].$ This is equivalent to having $\text{ord}_q(f_0) < \text{ord}_q(f_i)$ for each $i \geq 1$.

Since $q$ is not a base point of $\mathbf{M}$, this happens only if $\text{ord}_q(f_0) = -D(q)$ and $\text{ord}_q(f_i) \geq -D(q)$ for each $i \geq 1$.

This happens only if $f_0, \ldots, f_n$ is a basis for $L(D-q)$.

However, we choose the basis so it was a basis for $L(D-p)$ so $L(D-p) = L(D-q)$. Thus, every function $f \in L(D)$ that satisfies $\text{ord}_p(f) > -D(p)$ also satisfies $\text{ord}_q(f) > -D(q)$.

Thus, $L(D-p) = L(D-p-q)$ since $p \neq q$. Thus, $L(D-p) = L(D-p-q)$. This gives the 1st statement.

We have

$$l(D-p) = l(D)+1,$$

since $[D]$ is base point free. Thus, we have
\[ \ell(D-p-q) = \ell(D)-1 \quad \text{or} \quad \ell(D-p-q) = \ell(D)-2. \quad \text{However,} \\
\text{we have just seen if } \phi_D \text{ is 1-1 then } \ell(D-p-q) \leq \ell(D-p), \]
\[ \ell(D-p-q) = \ell(D)-2. \]

Conversely, if the dimension does always drop by 2 then
\[ \ell(D-p-q) \leq \ell(D-p) - \ell(D) \]

is distinct for every \( p \neq q \) so \( \phi_D \) is 1-1.

The next step is to determine when \( \phi_D \) is not only 1-1, but actually a holomorphic embedding.

**Lemma 5.21:** Let \( X \) be a compact R.S., \( D \in \text{Div}(X) \), 1D1 base point free.

Assume \( \phi_D \) is 1-1. Fix \( p \in X \). Then the image of \( \phi_D \) is a
holomorphically embedded R.S. means \( \phi_D(p) \) iff \( \ell(D-2p) = \ell(D-p) \).

**Proof:** Exercise. (Note if \( \text{f}_\ell(D-2p)-\ell(D-p) \), then \( \text{f}_\ell \) has exact
order \( -D(p)+1 \) at \( p \). Let \( f_0 \) have minimal order \( -D(p) \) at
\( p \). Then \( \text{f}_\ell/\text{f}_0 \) is a local coordinate on the image centered at \( p \).

We can rephrase in terms of the divisors again. We know
\( \ell(D-2p) \) has codim. 0 or 1 in \( \ell(D-p) \). We know \( \ell(D) \) is
base-point-free, then \( \ell(D-p) \) has codim. 1 in \( \ell(D) \). Thus,
we have the following:

Prop. 5.22: Let $X$ be a compact $\mathbb{C}$, $D \in \text{Div}(X)$, and $|D|$ base-point-free.

Then $\phi_D$ is a $1:1$ holomorphic map and an isomorphism onto its image (which is a holomorphically embedded $\mathbb{C} \times \mathbb{P}^n$)

iff for every $p, q \in X$ we have $\mathcal{L}(D-p-q) \cong \mathcal{L}(D) - 2$.

(Note we exclude $p = q$ here!)

Then this situation we think $\phi_D$ as just putting $X$ into $\mathbb{P}^n$.

Note the $n$ here is $\mathcal{L}(D)$.

We call a $D \in \text{Div}(X)$ so that $|D|$ is base-point-free and $\phi_D$ is an embedding a very ample divisor.

Exercise: 1) Let $D \in \text{Div}(\mathbb{C}^n)$ with $\deg D > 0$. Show $D$ is very ample.

2) Let $X$ be a complex torus. Let $D \in \text{Div}(X)$ with $\deg D > 3$.

Show $D$ is very ample.

Before we move back to applications of Riemann-Roch, let briefly recall hyperplane divisors and then relate them to linear systems.

Let $H \subseteq \mathbb{P}^n$ be a hyperplane, i.e., defined by the vanishing of
a degree 1 polynomial. We can, hence, define a divisor associated to $H$. We can then pull this back to a divisor on $X$. Explicitly, we have the following: let $\phi: X \to \mathbb{P}^n$ be holomorphic and let $p \in X$. Let $H$ be given by the homogeneous equation $L=0$. Choose another homogeneous linear equation $M$ so that $M$ does not vanish at $\phi(p)$. Let $h = (L/M) \circ \phi$. (Note $L/M$ defines the hyperplane division in $\mathbb{P}^n$, the $\phi$ is giving the pullback.) The function $h$ is defined near $p$. Write $\phi = [g_0(z) : \ldots : g_n(z)]$ for $g_i$ holomorphic near $p$, not all 0 at $z=0$. We set $\phi^*(H)(p)$ to be $\text{ord}_p(h)$. Note since $h$ is holomorphic near $p$, $\phi^*(H)(p) \geq 0$ and $\phi^*(H)(p) > 0$ iff $\phi(p) \in H$. We call $\phi^*(H)$ the hyperplane divisor for $\phi$.

**Exercise:** Check this is well-defined. (See the proof we gave before.)

**Lemma 5.27:** Let $\mathbb{P}^n$ have homogeneous coordinates $[x_0 : \ldots : x_n]$ and $H$ defined by $L = \sum a_i x_i = 0$. Let $\phi: X \to \mathbb{P}^n$ be defined by $\phi = [f_0 : \ldots : f_n]$ and set $D = \text{min}_i \text{div}(f_i)$. This
\( \phi^*(H) = \text{div} \left( \sum_i a_i f_i \right) + D. \)

Proof: Fix \( p \in X \) and choose \( j \) so that \( \text{ord}_p(f_j) = -D(p) \) is the minimum order. This gives \( x_j(p) \neq 0 \) so we can take \( M = x_j \) in the definition of \( \phi^*(H) \). We have
\[
h = \frac{\left( \sum_i a_i f_i \right)}{f_j},
\]
and this does not vanish identically near \( p \) since \( X \not\subset H \).

Thus,
\[
\text{ord}_p(h) = \text{ord}_p(\sum_i a_i f_i) - \text{ord}_p(f_j)
\]
\[
= \text{ord}_p(\sum_i a_i f_i) + D(p).
\]

Corl. 5.23: Let \( \phi: X \to \mathbb{P}^n \) be a holomorphic map. Then the set of hyperplane divisors \( \mathfrak{h} \phi^*(H) \) from the linear system \( H \).

Proof: This follows immediately from the previous lemma.

We may make use of Riemann-Roch to conclude powerful results on algebraic curves.
Proposition 5.24: Let $X$ be an algebraic curve of genus $g$. Then every $D \in \text{Div}(X)$ with $\deg D > 2g+1$ is very ample, i.e., the complete linear system $|D|$ has no base points and $\phi_D : X \to \mathbb{P}^n$ is a holomorphic embedding to a projective curve of degree $\deg D$.

Proof: Let $p, q \in X$. We need to show $l(D-p-q) = l(D)-2$. Since

$\deg D > 2g+1$, we have $\deg (D-p-q) > 2g+1 - 2 = 2g+1 > 2g+2$.

Thus, $l(K-D+p+q) = 0$. Thus,

$$l(D-p-q) = \deg(D-p-q) + 1 - g$$

$$= \deg D - 1 - g$$

and

$$l(D) = \deg D + 1 - g.$$  

Thus,

$$l(D) - l(D-p-q) = 2,$$

which gives the result. 

We are now in a position to prove every algebraic curve is projective, as was claimed earlier.

Theorem 5.25: Every algebraic curve can be holomorphically embedded.
into projective space.

**Proof:** This is essentially trivial now. Let \( p \in X \). Set \( D = (2g+1) \cdot p \).

Then by the previous result \( D \) is very ample and the associated map \( \phi_D \) is the required holomorphic embedding.

One actually gets a lot more.

**Cor. 5.26:** Let \( X \) be an algebraic curve and \( p \in X \). Then \( X - p \) can be embedded in affine space \( \mathbb{C}^n \).

**Proof:** Let \( p \in X \) and set \( D = (2g+1) \cdot p \). Recall that

\[ \phi_D \] is given as the set of hyperplane divisors. Thus,

there is a hyperplane \( H \) so that

\[ \phi_D^* (H) = (2g+1) \cdot p \cdot \]

This means the inverse image of \( H \) is just the point \( p \) as a set. Thus, \( X - p \) is embedded into \( \mathbb{P}^n - H \)

via \( \phi_D \). However, \( \mathbb{P}^n - H \cong \mathbb{C}^n \).

We now return to classifying some Riemann surfaces of small genus. Recall we saw all curves of genus \( 0 \) are
all isomorphic to \( \mathbb{C}^* \cong \mathbb{P}^1 \). We now consider curves of genus 2 again.

Let \( X \) be an algebraic curve of genus 2. Let \( D \in \text{Div}(X) \) with \( \deg D = 3 \). Then we have \( D \) is a very ample divisor as was given in a previous exercise. It also follows because \( \deg D = 3 \geq 2(1)+1-3 \). Riemann-Roch gives

\[
\ell(D) - \ell(K-D) = \deg D + 1 - 3
\]

\[\Rightarrow \]
\[\ell(D) = 3.\]

Thus, \( \Phi_D : X \rightarrow \mathbb{P}^2 \). Moreover, \( \deg D = 3 \) gives that the hyperplane divisor \( \mathcal{O} \) of degree 3 and so the image

is a smooth cubic curve. We have shown the following.

**Prop. 5.27:** Every algebraic curve of genus 1 is isomorphic to a smooth projective plane cubic.

We can also relate genus 2 curves to complex tori. We will come back to this when we do Abel's theorem.
We showed before every curve of genus 2 is hyperelliptic. We now move on to genus 3 and higher. We need the following result about the linear system of a canonical divisor. Let $K$ be a canonical divisor on $X$, i.e., $K = \text{div}(w)$ for $w$ a meromorphic 1-form on $X$. Recall, $\deg K = 2g - 2$. In particular, using Riemann-Roch, we have

$$
\ell(K) - \ell(0) = 2g - 2 + 1 - g
$$

$$
\ell(K) - 1 = g - 1,
$$

i.e.,

$$
\ell(K) = g.
$$

**Lemma 5.28:** The linear system $|K|$ on an algebraic curve $X$ of genus $g \geq 1$ is base-point-free.

**Proof:** Fix $p \in X$. We need to show $L(K-p) \neq L(K)$. We know from above that $\ell(K) - g$. Recall that we showed for any compact Riemann surface $X$ of genus at least 1 that $L(p) \cong \mathcal{O}$, i.e., $\ell(p) = 1$. Thus, R-R gives

$$
\ell(p) - \ell(K-p) = \deg p + 1 - g
$$

$$
= 2 - g
$$

i.e.,

$$
\ell(K-p) = \ell(p) - 2 + 3 = g - 1. \text{ Thus, } L(K-p) \notin L(K).
$$
Let $X$ be an algebraic curve of genus at least 3.

Let $K \in \text{KDiv}(X)$ and observe $|K|$ consists of the holomorphic 1-forms on $X$. We have just seen $K$ is base-point-free and $\ell(K) = g$, so we obtain a holomorphic map

$$\phi_K : X \to \mathbb{P}^{g-1}.$$ 

This is an incredibly important map; it is referred to as the canonical map for $X$. The basic question is when is $\phi_K$ an embedding?

We know $\phi_K$ is not an embedding if there are points $p, q \in X$ so that $\ell(K-p-q) \neq \ell(K) - 2 = g - 2$. Since $|K|$ has no base points, this would require $\ell(K-p-q) = g - 1$. We apply R-R to obtain

$$\ell(K-p-q) - \ell(K-K+p+q) = \deg(K-p-q) + 1 - g$$

i.e.,

$$\ell(K-p-q) = \ell(p+q) + 2g-2 - 2 + 1 - g = g - 3 + \ell(p+q).$$

Thus, $\phi_K$ fails to be an embedding if $\ell(p+q) = 2$. 

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Suppose \( \mu(p+q) = 2 \). Let \( f \in L(p+q) \), \( f \) not a constant.

Then \( f \) has a pole of order 1 at \( p \) and \( q \). Thus, the associated map \( F: X \to \mathbb{C}^2 \) has degree 2 since \( p \) and \( q \) both lie over \( \infty \). But no other points can. One has the following result. (Note \( p+q \) above gives a pole of order 2.)

**Prop. 5.29:** Let \( X \) be a compact R.S. Let \( F: X \to \mathbb{C}^2 \) be a degree 2 map. Then \( X \) is a hyperelliptic curve.

Using this we have:

**Prop. 5.30:** Let \( X \) be an algebraic curve of genus \( g \geq 3 \). Then

- \( \phi_k \) is an embedding iff \( X \) is not hyperelliptic.
- If \( X \) is not hyperelliptic, then \( \phi_k \) embeds \( X \) into \( \mathbb{P}^{g-1} \) as a smooth projective curve of degree \( 2g-2 \).

**Proof:** We saw above that if \( \phi_k \) is not an embedding then \( X \) is hyperelliptic. Suppose \( X \) is hyperelliptic.

Then there is a degree 2 mapping \( \pi: X \to \mathbb{C}^2 \).
We have the divisor \( \pi^* (\omega) \) has degree 2; write it as \( p+q \). (\( p \neq q \) is possible!) Then \( \lambda(p+q) = 2 \)
so \( \phi \) is not an embedding.

The result on the degree follows because \( \deg(X) = 2g - 2 \)
and it is a fact that in general one has

\[
\deg (\phi_p(X)) = \deg D.
\]

Before we can classify curves of genus 3, we need to study
the question of what hypersurfaces \( F=0 \) the image of \( X \) in
\( P^n \) can possibly lie on. Let \( D \in \text{Div}(X) \) be a very ample
divisor and \( \phi_p : X \to P^n \) the associated embedding. Let \( \mathcal{P}(n,k) \)
denote the vector space of homogeneous polynomials of degree \( k \)
in the \( n+1 \) homogeneous variables of \( P^n \). We have

\[
\dim \mathcal{P}(n,k) = \binom{n+k}{k}.
\]

Now we fix a \( k \) and a homogeneous polynomial \( F_0 \) of degree \( k \)
that does not vanish identically on \( X \). We consider the
intersection divisor \( \text{div}(F_0) \) on \( X \). Recall the hyperplane divisors
on $X$ are exactly the divisors in $\text{Id}_1$. Furthermore, recall that two intersection divisors of the same degree are linearly equivalent. Thus, if we choose the hyperplane divisor $H$ so that $\phi^*(H) = D$, then since $\text{div}(F_0) \sim K \cdot \phi^*(H)$, we have $\text{div}(F_0) \sim K \cdot D$.

Let $F$ be another homogeneous polynomial of degree $n$ so that $F = F/F_0$ is a meromorphic function on $X$. By construction we have the poles of $F$ are bounded by $\text{div}(F_0) \sim K \cdot D$. Thus, we have a $C$-linear map

$$R_k : \mathcal{P}(n, n) \to L(KD)$$

$$F \mapsto F/F_0.$$

This shows that $F \in \ker(R_k)$ iff $F$ vanishes identically on $X$.

Thus, $\ker(R_k)$ consists of the homogeneous polynomials giving hyperplanes in $\mathbb{P}^n$ containing $X$.

Observe that as $k$ grows, the dimension of $\mathcal{P}(n, n)$ grows like $k^n$. However, R-R gives for large $n$ that $L(KD) \sim (\deg D) \cdot K + 5$.

Thus, for large $n$ we will have $\ker(R_k)$ is large.

We now want to talk a bit about how surfaces in $\mathbb{P}^n$ an
Recall that $\mathbb{P}^n$ is an $n$-dimensional complex manifold. Thus, if we impose an equation $F = 0$, the complex manifold given by this should have dimension $n-2$. Thus, to get a Riemann surface in $\mathbb{P}^n$ we would want to intersect $n-1$ equations. We still need this to be smooth. We have already considered this for plane curves.

**Def:** Let $F_1, \ldots, F_{n-1}$ be $n-1$ homogeneous polynomials in $n+1$ variables $x_0, \ldots, x_n$. Let $X$ be the common zero locus in $\mathbb{P}^n$. We say $X$ is a smooth, complete intersection curve in $\mathbb{P}^n$ if the $(n-1) \times (n+1)$ matrix of partial derivatives $(\frac{\partial F_i}{\partial x_j})$ has rank $n-1$ at every point in $X$.

**Prop. 5.31:** A smooth complete intersection curve in $\mathbb{P}^n$ is a compact Riemann surface. Moreover, at every point of $X$ one can take as a local coordinate a ratio $X_i/X_j$ of the homogeneous coordinates.

**Proof:** Exercise. See implicit function theorem. 

It turns out not all Riemann surfaces in projective $n$-space are smooth complete intersection curves. For example, the twisted
Cubic in \( \mathbb{P}^3 \) given by
\[
X_0 X_3 = X_1 X_2 \\
X_0 X_2 = x_2 \\
X_1 X_3 = X_1^2
\]

is not a smooth complete intersection. Now at any particular point one only needs 2 of the 3 equations, but it won't work globally.

For example, mean \( f: 0; 0; 0; 0 \) the curve is cut out by
\[
X_0 X_3 = X_1 X_2 \\
X_0 X_2 = x_2
\]

Def: A **local complete intersection curve** in \( \mathbb{P}^n \) is a locus \( X \subset \mathbb{P}^n \) given by the vanishing of a set \( \{ F_0, \ldots, F_m \} \) of homogeneous polynomials such that at each \( p \in X \), \( X \) is described by \( n-1 \) of the polynomials
\[
F_{k_1} = F_{k_2} = \cdots = F_{k_{n-1}} = 0
\]

satisfying the non-singularity condition that the \( (n-1) \times (m-n) \) matrix of partial derivatives \( \frac{\partial F_{k_j}}{\partial X_j} \) has maximal rank \( n-1 \) at the point \( p \).

Prop. 5.32: Every connected local complete intersection curve \( X \) in \( \mathbb{P}^n \) is a compact Riemann surface. Moreover, at every point of \( X \) one can take as a local coordinate a ratio \( X_i / X_j \) of the homogeneous coordinates.
For a local complete intersection defined by the vanishing of certain polynomials $F_1, \ldots, F_k$, one obtains for free a bunch of equations that vanish on $X$, namely, any linear combination $\sum G_i F_i$ for example where the degree of $G_i$ is chosen so all the degrees of the $G_i F_i$ are the same. This is really the first step of showing all projective curves are local complete intersections.

Let $D$ be a specific divisor. We can use $R \cdot R$ to get quite a bit of information about $L(KD)$. We know $\dim \ker (R_0) = \dim P(n-1, 1) - L(KD)$.

Let $\deg D \geq g$ so $\deg (KD) \geq kg > 2g-1$ for $k \geq 2$. Thus,$$
L(K-\omega D) = 0.$$Thus, we have the following result.

**Lemma 5.3:** Let $D$ be a very ample divisor on an algebraic curve $X$ of degree at least $g$. Then for every $k \geq 2$, we have

$$\dim \ker (R_k) \geq \binom{n+k}{k} - \deg(D) \cdot k - 1 + g.$$We are now in a position to study curves of genus $3$. We will consider curves of genus $3$ and $4$ in this chapter. The same type of ideas can be used to study curves of higher genera, but the results are more complicated and less complete.
Let $X$ be a genus 3 curve that is not hyperelliptic. Then we have $\phi_k : X \to \mathbb{P}^2$ and the image of $X$ is a smooth curve of degree $\deg(k) = 2(3) - 2 = 4$. From the above analysis we have

$$\dim(\ker R_k) \geq \left(\binom{2+4}{4}\right) - 4(\deg(k) - 1) + 3 = 15 - 16 + 2 = 1.$$ 

Thus, there is a quartic polynomial that vanishes on $X$. (Note we identify $X$ with its image in $\mathbb{P}^2$.) Note this must be an irreducible polynomial for otherwise we would have a polynomial of degree less than 4 vanishing on $X$. Suppose there are two independent polynomials of degree 4 vanishing on $X$. The intersection then gives a finite set in $\mathbb{P}^2$. This can't be $X$, so it must be

$$\dim(\ker R_k) = 1.$$ 

Thus, $\ker(R_k)$ is generated by a single polynomial $F$ vanishing on $X$. We have any multiple $GF$ of $F$ also vanishes on $X$.

If $\deg F = k-4$, then $GF$ has degree $k$ so for any $k \geq 5$, we have a subspace of $\ker(R_k)$ of dimension

$$\dim \left( P(2, k-4) \right) = \frac{(k-2)(k-3)}{2}.$$ 

Note we have
\[ \dim \ker (R_k) \geq \binom{2 + k}{k} - (\deg K + 1 - 3) \]
\[ = \frac{(k+2)(k+1)}{2} - 4k + 2 \]
\[ = \frac{(k-2)(k-3)}{2} = \dim \mathbb{P}(2, k-4). \]

Next we see this is all of \( \ker (R_k) \). Let \( H \) be a polynomial that vanishes on \( X \) and is not a multiple of \( F \). Then with \( H \) and \( F \) have a common irreducible factor \((\#)\), or their intersection is a finite set \((\#)\). Thus every polynomial vanishing on \( X \) is a multiple of a quartic polynomial \( F \).

**Prop. 5.36:** Let \( X \) be an algebraic curve of genus 3. Then either
\( X \) is a hyperelliptic curve defined by an equation \( y^2 \in h(x) \)
with \( \deg h = 7 \) or \( 8 \), or the canonical map \( \Phi_k : X \to \mathbb{P}^2 \)
embeds \( X \) as a smooth plane quartic curve defined by the vanishing of a quartic polynomial.

We now move on to curves of genus 4. Let \( X \) be a curve of genus 4 that is not hyperelliptic so \( \Phi_k : X \to \mathbb{P}^3 \)
with the image having degree 6. We have

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\[
\dim \ker (R_2) \geq \binom{5}{2} - 2 \cdot 2 = 0
\]

\[
\dim \ker (R_3) \geq \binom{6}{3} - 3 \cdot 6 - 1 + 4 = 5
\]

The first equation gives that $5$ is a quadratic polynomial $F$ that vanishes on $X$. We claim $F$ is unique up to constant multiple. Suppose there are two such quadratics. Choose a general hyperplane $H \cong \mathbb{P}^2$.

Restrict $F$ and $F_1$ to $H$. Using Bezout's Theorem, we see the common zeros of $F$ and $F_1$ in $H$ consist of at most 4 points. However, $X$ intersecting with $H$ contains 6 points, so $X$ cannot be contained in the zero set of $F$ and $F_1$. Thus, $F$ is unique up to a scalar.

Let $[x:y:z:w]$ be the variables of $\mathbb{P}^3$. Since $F$ vanishes in $X$, so do the cubics $xF, yF, zF$, and $wF$.

Thus, for any linear polynomial $L$ we have $LF$ vanishes on $X$ and so lies in $\ker R_3$. This gives a 4-dimensional subspace in $\ker R_3$. Since $\dim \ker R_3 > 5$, there is a cubic $G$ that is not a multiple of $F$ that vanishes on $X$.

We again intersect with $H$ to see $F$ and $G$ have 6 zeroes on $H$; since $X$ intersects $H$ in 6 points and
X is contained in the common zeroes of F and G, we see on H that X is the common zero set of F and G. In fact, if one is more careful one can show this same result in the case H intersects X in less than 6 points. This more general case gives the following.

Prop. 5.37: Let X be an algebraic curve of genus 4. Then

either X is hyperelliptic defined by \( y^2 = h(x) \) with \( \deg h = 9 \) or 10,

or X embeds X in \( \mathbb{P}^3 \) as a smooth curve of degree 6

defined by the vanishing of a quadratic and cubic polynomial. In fact, X is the complete intersection of F and G.