In this chapter, we will prove the Riemann-Roch theorem stated in the last chapter. There are certainly short proofs that work for algebraic curves over general fields using sheaf cohomology. However, the proof we will give is essentially the hands-on version of this proof, so we will be doing the explicit computations that are behind the cohomology arguments. This will help in understanding the more general proof when you see it in algebraic geometry.

Throughout this chapter, we let \( X \) be a compact Riemann surface.

For each point \( p \in X \), we fix a local coordinate \( z_p \) centered at \( p \).

**Def:** A Laurent tail divisor on \( X \) is a finite formal sum

\[
\sum_{p} r_p(z_p) \cdot p
\]

where \( r_p(z_p) \) is a Laurent polynomial in the coordinate \( z_p \), i.e., a Laurent series with finitely many terms.
Note this is not a divisor since \( R_p(z) \) is not in general an integer. However, the collection of Laurent tail divisors does form a group under formal addition and is denoted \( \Upsilon(X) \). In fact, it form a sheaf but we won't really need that here. We do make use of divisors to put conditions on the Laurent tail divisors. Given \( D = \text{Div} (X) \), consider

\[
\Upsilon[p](X) = \left\{ \sum r_p \cdot p : \text{for all } p \in R_p \text{ the top term of } r_p \text{ has degree strictly less than } -D(p) \right\}.
\]

**Example:** Let \( D = 0 \). Then \( \Upsilon[0](X) \) is the group of Laurent tail divisors \( \sum r_p \cdot p \) so that each monomial term \( r_p \) has every term in \( r_p \) has strictly negative degree.

There is a natural truncation map \( \Upsilon(X) \rightarrow \Upsilon[p](X) \)
given as follows: for each monomial \( r_p \), the truncation map removes all terms in \( r_p \) of degree \(-D(p)\) and higher. More generally, given divisors \( D_1 \) and \( D_2 \) with \( D_1 \leq D_2 \), we have a truncation map.
\[ \tilde{t}^{D_1}_{D_2} : T[D_1](x) \rightarrow T[D_2](x) \]

defined by removing from each \( r_p \) all terms of degree \(-D_2(p)\) and higher. For example, if \( D_1 = 2 \cdot p \) and \( D_2 = 3 \cdot p \), then \((Z_p^{-4} + Z_p^{-3}) \cdot p \in T[D_1](x)\), and then

\[ \tilde{t}^{D_1}_{D_2}((Z_p^{-4} + Z_p^{-3}) \cdot p) = Z_p^{-5} \cdot p \in T[D_2](x). \]

One has the truncated Laurent tail divisors can be realized as a sheaf, and one the truncation map is a sheaf map.

Continuing with defining useful maps, given \( f + M_X(x) \) and \( D \in \text{Div}(x) \), we have a multiplication map given by

\[ \mu^D_f : T[D](x) \rightarrow T[D - d \cdot (f)](x) \]

given by sending \( \Sigma r_p \cdot p \) to the truncation of \( \Sigma (f \cdot p) \cdot p \).

We have \( \mu^D_f \) in an isomorphism with inverse given by

\[ \mu^{D - d \cdot (f)}_{1/f} \] (check as an exercise!)

We also have a Laurent tail version of the division function. For \( D \in \text{Div}(x) \). The map \( \alpha_D \) is then given by

\[ \alpha_D : M(x) \rightarrow T[D](x) \]
defined by sending \( f \) to \( \Sigma r_p \cdot p \) where \( r_p \) is the truncation of

the Laurent series \( f(z_p) \) of \( f \) in terms of \( z_p \) when we remove all terms

of order \(-D_p\) and higher. This map is of fundamental importance
to what follows.

The map \( \alpha_D \) and the truncation map \( t_D^D \) interact

nicely. Namely, let \( D_1, D_2 \in \text{Div}(X) \) with \( D_1 \leq D_2 \). Then

we have

\[
\begin{align*}
\mathcal{M}(X) & \rightarrow \mathcal{T}(D_1)(X) \rightarrow \mathcal{T}(D_2)(X) \\
\alpha_D^{D_1} & \rightarrow t_D^{D_2} \circ \alpha_D^{D_1}
\end{align*}
\]

i.e.,

\[ \alpha_D^{D_2} = t_D^{D_1} \circ \alpha_D^{D_1}. \]

One should check this as an

exercise. We also have \( \alpha_D \) is compatible with multiplication,

namely, if \( f, g \in \mathcal{M}(X) \), then

\[ \mu f (\alpha_D(g)) = \alpha_D(\text{div}(f))(f) \]

for any \( D \in \text{Div}(X) \).
Let $D$ be a divisor so that $D(p) = 0$. Then $\alpha_D(h)$ has

Laurent polynomial at $p$ with only negative terms, i.e., the strictly
negative term in the Laurent series of $f$ at $p$. Thus,

the $p^{th}$ term of $\alpha_D(h)$ is 0 if $f$ is holomorphic at $p$.

We can interpret the Riemann–Roch space in this language

as well (which we better be able to do if this is useful for the proof

of Riemann–Roch!) We know if $f \in L(D)$, then the poles of

$f$ have order no worse than $-D(p)$ at each $p$. Thus

saying upon truncating at the $-D(p)$ level and higher, we

get 0 at every point. Thus,

$$L(D) = \ker(\alpha_D).$$

This is extremely important to what is upcoming, which is the
development of cohomology (in secret!)

The question we ask is, if we have an element of

$T(D)(x)$, is it in the image of $\alpha_D$? In other words, given

a Laurent tail, is there a global meromorphic function with
These tails at each point.

Let \( Z \in \mathcal{T}_D(x) \). Then for all but finitely many \( p \) we will have \( r_p = 0 \) and \( D(p) = 0 \). Thus, the preimage of \( Z \) under \( \alpha_0 \) would have to satisfy being holomorphic at \( p \). Thus, we are looking for a function that \( \alpha_0 \) is holomorphic at all but finitely many points, but has prescribed divisors at the other points.

We state the problem in terms of the cohomology of \( \alpha_0 \), namely, we want to determine

\[
H^1(D) = \text{coh}_{\text{eq}}(\alpha_0) = \frac{\mathcal{T}_D(x)}{\text{Im}(\alpha_0)}.
\]

Thus, we see \( Z \in \mathcal{T}_D(x) \) is in the image of \( \alpha_0 \) iff it is 0 in \( H^1(D) \). Thus, \( H^1(D) \) is measuring the failure of being able to find a function with prescribed tails at finitely many points that are holomorphic elsewhere.

For those with some algebraic geometry background,

\[
H^1(D) = H^1(X_{\text{zar}}, \mathcal{O}_{X, \text{alg}}[D]).
\]
Note from the definitions, we have an exact sequence

\[ 0 \rightarrow L(D) \rightarrow M(x) \xrightarrow{\delta_D} T[D](x) \rightarrow H^1(D) \rightarrow 0, \]

i.e.,

\[ 0 \rightarrow M(x)/L(D) \xrightarrow{\delta_D} T[D](x) \rightarrow H^1(D) \rightarrow 0. \]

It will be important to be able to compare \( H^1(D) \) for various divisors. Let \( D_1, D_2 \in \text{Div}(X) \) with \( D_1 \leq D_2 \). Thus we have \( L(D_1) \leq L(D_2) \) and the truncation map

\[ t : T[D_1](x) \rightarrow T[D_2](x) \]

is defined. We recall \( \delta_D \) commute with truncation, so we have the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M(x)/L(D_1) \\
\downarrow & & \downarrow t \\
0 & \rightarrow & M(x)/L(D_2)
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& & \downarrow \\
& & H^1(D_1) \\
& & \downarrow \\
& & H^1(D_2)
\end{array}
\]

We analyze the kernels of the vertical arrows. The \( \delta_{D_1} \)
on is easy, the kernel is \( L(D_2)/L(D_1) \). Thus,
Claim \( \dim \left( \ker \left( \lambda \left( \frac{M(x)}{L(D)} \right) \cong \frac{M(x)}{L(D_1)} \right) \right) = \dim L(D_2) - \dim L(D_1) \).

Next we deal with the second vertical map; namely we consider the kernel of the truncation map \( t : T(D_j)(x) \rightarrow T(D_k)(x) \).

The elements in this kernel are those Laurent tail division \( \sum_{p} z_p \ p \) so that the top term has order less than \(-D_1(p)\) and the bottom term has order at least \(-D_2(p)\). Thus, there are \( D_2(p) - D_1(p) \) possible monomials that can occur in the kernel, namely those \( z_p^k \) with \(-D_2(p) < k < D_1(p)\). Note the conditions at each \( p \) are independent, so the dimension of the kernel of \( t \) is \( \sum_{p} (D_2(p) - D_1(p)) \), i.e.,

\[
\dim \left( \ker \left( t : T(D_j)(x) \rightarrow T(D_k)(x) \right) \right) = \deg D_2 - \deg D_1.
\]

Write \( H'(D_j/D_k) \) for the kernel of the induced map from \( H'(D_1) \) to \( H'(D_2) \). The snake lemma gives an exact sequence

\[
0 \rightarrow L(D_j)/L(D_k) \rightarrow \ker(t) \rightarrow H'(D_j/D_k) \rightarrow 0.
\]

Since \( \dim (\ker(t)) \) is finite dimensional, we have...
\[
\dim H'(D_1/D_2) \leq 0 \text{ as well. Using the computations above we have:}
\]

Prop. 6.1: Let \( D_1, D_2 \in \text{Div}(X) \), \( D_1 \leq D_2 \). Then

\[
\dim H'(D_1/D_2) = (\deg D_2 - \dim L(D_2)) - (\deg D_1 - \dim L(D_1)).
\]

One should also note that we can extend the sequence of kernels into a long exact sequence

\[
0 \to L(D_1) \to L(D_2) \to \ker (\cdot) \cong \mathbb{C}^{\deg (D_2 - D_1)} \to H'(D_1) \to H'(D_2) \to 0.
\]

This will be useful shortly.

We now show that \( H'(D) \) is actually a finite dimensional \( \mathbb{C} \)-vector space. We have just seen that if \( D_1 \leq D_2 \), then \( H'(D_1/D_2) = \ker (H'(D_1) \to H'(D_2)) \) is finite dimensional. This is measuring the difference between \( H'(D_1) \) and \( H'(D_2) \) since we have \( H'(D_1) \to H'(D_2) \). This, we really only need to show finite dimensionality for one, then we will have it for the other.

We will show it is finite dimensional for a particular divisor and deduce this gives it for all divisors.
Lemma 6.2: Let $f \in M(x, t) \subset C$. Let $D = \text{div}(f)$, the divisor of poles of $f$. Then for large $m$ we have $\dim H'(O_{md})$ is independent of $m$.

Proof: We apply Prop. 6.1 with $D_1 = 0$ and $D_2 = md$. This gives

$$\dim H'(O_{md}) = (m \deg D - \dim L(md)) - (0 - 1) = m \deg D - \dim L(md) + 1.$$  

We saw before that the transcendence degree of $M(x)$ over $C$ has degree 1. In fact, one can show (see Miranda) that for $f \in M(x) \setminus C$, we have $[M(x): C(f)] = \deg D$ for $D = \text{div}(f)$. Moreover, using Lemma 1.20 of Miranda we have there is an $m_0 \in \mathbb{Z}$ such that

$$\dim L(md) \geq (m - m_0 + 1) \deg D$$

for all $m > m_0$. We apply this to the above formula to obtain

$$\dim H'(O_{md}) \leq m \deg D - (m - m_0 + 1) \deg D + 1 = 1 + \deg D (m_0 - 1).$$

This is independent of $m$, so we have an upper bound.
on $H'(0^mD)$ independent of $m$.

Now if $0 < m < m_2$ we have $0 < mD < m_2D$, so $H'(0) \to H'(mD)$ and $H'(m_2D) \to H'(m_2D)$. Composing there is the surjection from $H'(0)$ to $H'(m_2D)$. Thus, $H'(0^mD) \subseteq H'(0^m_2D)$. Thus, as $m$ increases $\dim H'(0^mD)$ in a monotonically increasing sequence that is bounded above, so it eventually stabilizes.

Thus, given $f \in H'(X) - C$ and $D = \text{div}(f)$, we have there is a constant $M$ (depending only on $D$) so that

$$\deg (mD) - \dim L(mD) \leq M$$

for all $m \geq 0$ since this is $\dim H'(0^mD) - 1$. (This stabilizes.

We just take $M$ to be the max of $\dim H'(0^mD) - 1$. This is actually true for all $D \in \text{Div}(X)$.

**Lemma 6.3**: Let $X$ be an algebraic curve. There exist $M \in \mathbb{Z}$ so that

$$\deg E - \dim L(E) \leq M$$

for all $E \in \text{Div}(X)$. 

\(\quad \Box\)
Before we can prove this, we require the following lemma.

Lemma 6.4: Let $E$ be a divisor on a compact Riemann surface $X$.
Let $f \in \mathcal{M}(X)$ and set $D = \text{div}_{\infty}(f)$. There is an integer $m > 0$ and a meromorphic function $g$ on $X$ such that $E - \text{div}(g) \leq mD$.
Moreover, we can take $g$ to be a polynomial in $f$, i.e.,
\[ g = r(f) \quad \text{for some} \quad r(t) \in \mathbb{C}[t].\]

Proof: Let $p_1, \ldots, p_i$ be the points in \text{supp}(E) that are not poles of $f$ and for which $E(p_i) > 1$. We have $f(p_i) \in \mathbb{C}$ and so $f-f(p_i)$ has a zero at $p_i$ and has the same poles as $f$. Thus, \((f-f(p_i))^{E(p_i)}\) has a zero at $p_i$ of order at least $E(p_i)$, and \((f-f(p_i))^{E(p_i)}\) has no poles other than those of $f$. We take the product over all $i$. This gives a meromorphic function $g$ that is a polynomial in $f$ such that $E - \text{div}(g)$ is positive only at the poles of $f$. Thus, for some integer $m$,
\[ E - \text{div}(g) \leq mD,\]
where $D = \text{div}_{\infty}(f)$.

We can now prove Lemma 6.3.
Proof: (Lemma 6.3): Fix \( f \in M_1(X) \) and let \( D = \text{div}(f) \). Let

\[ M \text{ be such that} \]

\[ \deg(mD) - \dim L(mD) \leq M \]

for all \( m \geq 0 \). Now let \( E \in \text{Div}(X) \). The previous lemma gives \( g + M(X) \) and an integer, \( m \), s.t. \( E = E \cdot \text{div}(g) + mD \).

We have \( \deg E' = \deg E \) and \( L(E') \cong L(E) \). Thus,

\[ \deg E - \dim L(E) = \deg (E') - \dim L(E') \]

This gives

\[ \deg E - \dim L(E) = \deg E' - \dim L(E') \]

\[ = (\deg(mD) - \dim L(mD)) - \dim H'(E'/mD) \]

\[ \leq \deg(mD) - \dim L(mD) \]

\[ \leq M. \]

This gives the result. \( \Box \)

This gives a divisor \( E_0 \) so that \( \deg E_0 - \dim L(E_0) \) is maximal.

Lemma 6.5: For the divisor \( E_0 \), we have \( H'(E_0) = 0 \).

Proof: Suppose \( H'(E_0) \neq 0 \). This means there exists \( Z \in \mathbb{T}(E_0)(X) \) so that \( Z \) is not in the image of \( \phi_{E_0} \). If we
increase $E_0 \to E$, we can truncate $Z \to 0$, i.e.,
$t(Z) = 0$ in $\Omega(E)(X)$. Thus, $t(Z) = 0$ in $H'(E)$. This gives $Z \in H'(E_0/E)$, so $H'(E_0/E) \neq 0$.

However, this gives

$$1 \leq \dim H'(E_0/E) = (\deg E - \dim L(E)) - (\deg E_0 - \dim L(E_0)).$$

However, this cannot be positive since $\deg E_0 - \dim L(E_0)$ is maximal. This gives the lemma.

We can now easily prove $\dim H'(D)$ is finite dimensional for any $D \in \text{Div}(X)$.

**Theorem 6.6:** Let $D \in \text{Div}(X)$. Then $H'(D)$ is a finite dimensional vector space.

**Proof:** Let $E_0$ be as above and write $D - E_0 = P - N$ with $P$ and $N$ non-negative divisors. Then $H'(E_0)$ surjects onto $H'(E_0 + P)$. Thus, $H'(E_0 + P) = 0$ as well. Thus, we have

$$H'(E_0 + P - N) \to H'(E_0 + P) = 0,$$

so

$$H'(E_0 + P - N) \cong H'(E_0 + P - N/E_0 + P).$$
i.e., we obtain $H'(D) = H'(E_0 + P - N)$ is isomorphic to the finite dimensional vector space $H'(E_0 + P - N / E_0 + P)$.

We can now give the first form of the Riemann-Roch theorem, namely, we have the statement in terms of cohomology.

**Theorem 6.7:** Let $X$ be an algebraic curve and $D \in \text{Div}(X)$.

Then

$$\dim L(D) - \dim H'(D) = \deg D + 1 - \dim H'(D).$$

**Proof:** We have the finite dimensionality of $H'(D)$ along with the fact that $H'(D_i) \rightarrow H'(D_0) \rightarrow 0$, giving

$$\dim H'(D_i / D_0) = \dim H'(D_i) - \dim H'(D_0).$$

for all $D_1, D_2$. Thus, we have

$$\dim H'(D_1) - \dim H'(D_2) = (\deg D_1 - \dim L(D_1))$$

$$- (\deg D_2 - \dim H'(D_2)).$$

Rearranging, we obtain

$$\dim L(D_1) - \deg D_1 - \dim H'(D_1) = \dim L(D_2) - \deg D_2 - \dim H'(D_2).$$
Since any two divisors have a common maximum, we find
\[ \dim L(D) = \deg D + \dim H'(D) \]
in constant over all divisors \( D \). If \( D = 0 \), we have this is \( 1 - \dim H'(D) \). This gives the result. \[ \square \]

The problem now is to relate this to the form of Riemann-Roch we already gave. This will come via duality, which relates \( H'(D) \) to \( L(k \cdot D) \). As far we have seen the \( H'(D) \) is a way to measure whether a Laurent tail divisor \( Z \) is the truncation of a meromorphic function on \( X \). In order to get the useful version of Riemann-Roch of the last chapter, we will need to compute this in another way. We will do this via the Residue Theorem. For example, let \( Z \in \mathcal{O}(X) \) with \( Z = \sum r_p \cdot p \).

Then when we ask if \( Z = \sigma_0 \) for some \( f \in \mathcal{M}(X) \), we are asking if there is a meromorphic function \( f \in \mathcal{M}(X) \) so that at each point \( p \) the negative terms in the Laurent series for \( f \) at \( p \) are exactly \( r_p \). Thus, we are trying to
specify the Laurent tails at all the poles of $f$.

Suppose we have such an $f$, and moreover suppose we
have a holomorphic 1-form on $X$. Then $f \omega$ has poles only
at the poles of $f$ and the negative terms in the Laurent
expansion for the 1-form $f \omega$ are determined by the negative
terms of $f$ and the form $\omega$. Thus, the negative terms
for $f \omega$ at $p$ are just the negative terms in the Laurent
series for $f \omega$ if $\alpha_0 (f) = \sum \rho \cdot p$. We now apply the
Residue Theorem to obtain $\sum \rho \cdot \text{Res}_p (f \omega) \cdot \rho = 0$, i.e., in this case we have
$\sum \rho \cdot \text{Res}_p (f \omega) = 0$, if $\alpha_0 (f) = \sum \rho \cdot p$. Thus, $\sum \rho \cdot \text{Res}_p (f \omega) = 0$

a necessary condition on $\omega$ for $\omega \in \text{Im } \alpha_0$. We will see that
one can essentially generalize this condition to any divisor (not
just $0$) and this will identify the space of 1-forms with
the dual of $H^1 (\mathcal{O})$. This is referred to as Serre duality.

Let $\text{Div} (X)$ and let $\omega \in L^1 (-D) \cong L (X - D)$. $\omega$
Then we have $\text{div} (\omega) = D$, i.e., $\text{ord}_p (\omega) = D (p)$ for all $p \in X$. 

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Thus, we can write
\[ \omega = \left( \sum_{n = D(p)}^{\infty} c_n z_p^n \right) dz_p \]

where we still use \( z_p \) as the local coordinates at \( p \).

Let \( f \in \Omega(\mathbb{C}) \) and write \( f = \sum_k a_k \overline{z}_p^k \). We compute

the residue of \( f \omega \) at \( p \):

\[ \text{Res}_p(f \omega) = \text{Coeff} \ of \ \overline{z}_p^1 \ in \ \left( \sum_k a_k \overline{z}_p^k \cdot \sum_{n = D(p)}^{\infty} c_n z_p^n \right) dz_p \]

\[ = \sum_{n = D(p)}^{\infty} c_n a_{-1-n} \, z_p^{-1-n} \, dz_p \]

This shows the residue only depends on those coefficients \( a_k \) of \( f \) with \( k < -D(p) \). In other words, the residue depends only on \( \alpha_D(f) \).

We use this to define, for a fixed \( \omega \), a residue map

\[ \text{Res}_\omega : \Omega(\mathbb{C}) \to \mathbb{C} \]

\[ \sum_{p \in D} r_p \cdot \omega \mapsto \sum_p \text{Res}_p (r_p \omega) \, . \]

We have just shown that

\[ \text{Res}_\omega (\alpha_D(f)) = \sum_p \text{Res}_p(f \omega) \, . \]

The Residue theorem gives

\[ \sum_p \text{Res}_p(f \omega) = 0 \, . \]
So \( i \in \ker(P_{\text{res}}) \). Thus, we have a map

\[ \text{Res}_{\text{d}} : H'(D) \rightarrow \mathbb{C}, \]

i.e., for each \( \omega \in \mathbb{L}^n(D) \) we have an element \( \text{Res}_{\text{d}} \omega \in H'(D)' \).

It turns out this is an isomorphism:

**Theorem 6.8 (Green's duality)**: Let \( D \in \text{Div}(X) \). The map

\[ \text{Res}_{\text{d}} : L^n(-D) \rightarrow H'(D)' \]

is an isomorphism of vector spaces. In particular, for \( K \in K\text{Div}(X) \), we have

\[ \dim H'(D) = \dim L^n(-D) = \mathcal{E}(K-D). \]

The proof of this will take several steps.

**Lemma 6.9**: The map \( \text{Res}_{\text{d}} : L^n(-D) \rightarrow H'(D)' \) is an injective linear map.

**Proof**: The fact this is a linear map is clear. The content here is to show it is injective. Let \( \omega \in L^n(-D) \), \( \omega \neq 0 \)

so that \( \text{Res}_{\text{d}} \omega \neq 0 \) in \( H'(D)' \). Thus, for
every \[ \sum_{p} \mathcal{R}_{p}(r_{p} \cdot w) = 0. \]

Fix \( p \) with local coordinate \( z_{p} \). Since \( w \in L^{(1)}(\mathbb{D}) \), we have \( a_{p}(w) \in D(\mathbb{D}) \). Write \( k = a_{p}(w) \), so \(-k \in -D(\mathbb{D})\) and \[ z_{p}^{-1} \cdot p \in \Omega(\mathbb{D}). \] Write \( w = \left( \sum_{n=1}^{\infty} c_{n} z_{p}^{n} \right) dz_{p} \) with \( c_{n} \neq 0 \). Then,

\[ \Re_{0}(z_{p}^{-1-k} \cdot p) = \Re_{p}(z_{p}^{-1-k} \sum_{n=1}^{\infty} c_{n} z_{p}^{n} dz_{p}) = c_{n} \neq 0. \]

This contradiction gives that \( \Re_{0} \) is an injection.

At this point we have

\[ L^{(1)}(\mathbb{D}) \subseteq L^{(k-D)} \hookrightarrow H^{(D)} \]

so

\[ L^{(k-D)} \subseteq \text{ann} H^{(D)} \]

We still need to show \( \Re_{0} \) is surjective. This requires two lemmas. First, note that we can identify \( H^{(D)} \) with

the space of linear functionals on \( \Omega(\mathbb{D}) \) that vanish on \( \mathcal{K}_{0}(M(\mathbb{D})) \). (Check this as an exercise.) This allows us to
do our computations with $\sigma_T(\mathcal{D}(x))$ instead of $\mathcal{H}(x)$, which is easier.

Let $\phi \in \sigma_T(\mathcal{D}(x))$ and suppose $\phi$ vanishes on $x^0(\mathcal{H}(x))$.

Let $f(x)$. Then we have

$$\phi \circ f : \sigma_T(\mathcal{D} + \text{div}(x))(x) \to C,$$

and

$$\phi \circ f \in \sigma_T(\mathcal{D} + \text{div}(x))(x),$$

and $x^0(\mathcal{D} + \text{div}(x)(x)) \subseteq \ker \phi \circ f.$

$$\phi \left( \mu_f \left( x^0 + \text{div}(x) \right) \right) = \phi \left( x^0 \left( f(x) \right) \right) = 0.$$

---

**Lemma 6.10:** Let $A \in \text{Div}(x)$ and let $\phi_1, \phi_2 \in \mathcal{H}(A)^\times$. Then there is a positive divisor $C \in \text{Div}(x)$ and $f_1, f_2 \in L(C)$

such that the two maps:

$$\mu_{f_1} \circ \sigma_T(A - C - \text{div}(x_1))(x) \to \sigma_T(A)(x) \to C,$$

$$\mu_{f_2} \circ \sigma_T(A - C - \text{div}(x_2))(x) \to \sigma_T(A)(x) \to C.$$

are equal.

**Proof:** Suppose we cannot find such a $C$ and $f_1, f_2$. This implies that for every positive divisor $C$ and every $f_1, f_2 \in L(C)$.
the map

\[ L(C) \times L(C) \to H^1(A - C) \wedge \]

\[ (f_1, f_2) \mapsto \int_X \chi_{A - C \cdot \text{div}(h)} \circ f_1 - \phi_2 \circ f_2 \circ \text{div}(\xi) \]

is injective. Thus, for every positive \( C \) we must have

\[ \dim H^1(A - C) \geq 2 \dim L(C). \]

For \( C \) large and positive, we apply Riemann-Roch (cohomological) to \( A - C \) to obtain

\[ \dim H^1(A - C) = \ell(A - C) - \deg(A - C) - 1 + \dim H^0(0) \leq \ell(A) - \deg A - 1 + \dim H^0(0) + \deg C. \]

For fixed \( A \), this grows like \( a + \deg C \) for a constant.

We now apply R-R to \( C \) to obtain

\[ \ell(C) \geq \deg C + 1 - \dim H^0(0). \]

Thus, \( 2 \ell(C) \) grows like \( b + 2 \deg C \). This is impossible, so we have our contradiction.

**Lemma 6.11:** Let \( D_1 \in \text{Div}(X) \) with \( w \in L^{(n)}(D_1) \) so that

\[ \text{Res}_{w, 0} : T_1[D_1](X) \to \mathbb{C} \text{ is well-defined. Assume} \]

\[ D_2 \geq D_1 \text{ and } \text{Res}_{w, 0} \text{ vanishes on ker } \epsilon_{D_2}^{D_1}. \text{ Then} \]
Proof: Suppose \( u \in L^{(1)}(-D_2) \). Then there exist \( p \in \mathbb{X} \) s.t.

\[
k = \text{ord}_p(w) < D_2(p).
\]

Consider the Laurent pair \( Z = z_p^{-k} \cdot p \). Then \( Z \in \text{ker}(\mathcal{T}) \), but \( \text{Re}_w(Z) \neq 0 \). This contradiction gives the lemma.

Exercise: Let \( f \in M(x) \) and \( u \in L^{(1)}(-D) \). Then show

\[
f \ast u \in L^{(1)}(-D - \text{div}(f))
\]

and

\[
\text{Re}_w(f \ast u) = \text{Re}_w f
\]

as linear functionals on \( \mathcal{F}(D + \text{div}(f)) \).

To finish the proof of dense duality, it only remains to show

\[
\text{Re}_w : L^{(1)}(-D) \to H^1(D)^\vee
\]

is surjective. We now give the proof.

Proof (Thm 6.8): Fix \( f \in \text{Div}(x) \) and \( \psi \in H^1(D)^\vee \). We consider

\[
\psi
\]

as a functional on \( \mathcal{F}(D)(x) \) that vanishes on \( \text{div} \) \( (M(x)) \).

Let \( w \in H^{(1)}(x) \), set \( K = \text{div}(w) \). Pick \( A \in \text{Div}(x) \) so that \( A \leq D \) and \( A \leq K \). This gives \( A \in L^{(1)}(-A) \), so \( \text{Re}_w \) is well-defined on \( \mathcal{T}[A](x) \). Let
\( \phi_A = \phi \circ t_A : \text{TA}(X) \to C \). Then \( \phi_A \) and Reo are both in \( \text{TA}(X)^{\nu} \). Thus, Lemma 6.4 gives a positive divisor \( C \in \text{Div}(X) \) and \( f_1, f_2 \in \mathcal{L}(C) \) so that

\[
\phi_A \circ t_A \circ \mu_{f_1} = \text{Reo} \circ t_A \circ \mu_{f_2}
\]

as elements of \( H'(A-C)^{\nu} \). The functional \( \text{Reo} \circ t_A \circ \mu_{f_1} \) is exactly the functional \( \text{Reo} \) acting on \( \text{TA}(A-C)^{\nu}(X) \). and \( \text{Reo} \circ \mu_{f_2} = \text{Reo}_{f_2} w \) on \( \text{TA}(A-C)^{\nu}(X) \). Thus, we have

\[
\phi_A \circ t_A \circ \mu_{f_1} = \text{Reo}_{f_2} w
\]

as elements of \( \text{TA}(A-C)^{\nu}(X) \). We compare with \( \mu_{f_1} \)

to obtain

\[
\phi_A = \phi \circ t_A = \text{Reo}(f_2/f_1) w
\]

as elements of \( \text{TA}(A-C)^{\nu}(X) \). Observe that

\[
(f_2/f_1) w \in L^{1/(n-1)}(-A+C+\text{div}(f_1))
\]

Moreover, we have that

\[
\text{Reo}(f_2/f_1) w \text{ must vanish on } \text{ker } \phi \circ t_A \circ \mu_{f_1}
\]

Thus

allows us to apply Lemma 6.10 to conclude that

\[
(f_2/f_1) w \in L^{1/(n-1)}(-A), \quad \text{and so } \phi_A = \text{Reo}(f_2/f_1) w\]

We recall
\[ \phi * \psi = \phi \circ \tau^A, \quad \text{as } \tau(\Phi^0_t)w \text{ vanishes on } \ker \tau^A. \]

Thus, \((\tau^2 \tau^1)_w \in L^{(1)}(-D)\) and \(\Phi = \tau(\Phi^0_t)w\). Thus, he map is surjective.

We can now rewrite Riemann-Roch as

\[ \ell(D) - \ell(k-D) = \deg D + 1 - \dim H^1(\mathcal{O}). \]

It only remains to calculate \(\dim H^1(\mathcal{O})\). Recall that \(\deg k = 2g - 2\). We can apply semi-continuity to the divisor \(k\) to obtain

\[ L^{(1)}(-k) \cong H^1(k)^\vee, \]

so

\[ \dim H^1(k) = \dim L^{(1)}(-k) = \dim L(k-k) = 1. \]

Thus,

\[ \dim H^1(k) = 1. \]

Moreover, we have

\[ L^{(1)}(0) \cong H^1(0)^\vee \]

is

\[ L(k). \]

Thus,

\[ \dim L(k) = \dim H^1(0). \]
Thus, applying Riemann-Roch (cohomological) to $K$:

$$l(K) - l(\omega) = \deg K + 1 - \dim H^0(\omega)$$

i.e.,

$$l(K) - 1 = 2g - 2 + 1 - \dim H^0(\omega)$$

i.e.,

$$\dim H^0(\omega) + \dim H^0(\omega) = 2g$$

i.e.,

$$\dim H^0(\omega) = g.$$

Note we calculated before that $l(K) = g$, but we used R-R. to do this so we can't use it here.

This result shows the topological genus, $g$, is equal to the arithmetic genus, $\dim H^0(\omega)$. We thus obtain the version of Riemann-Roch given in the last chapter. We also have $g = \dim L^1(\omega) = \dim \mathcal{L}^1(\omega)$. The dimension $\dim \mathcal{L}^1(\omega)$ is often referred to as the analytic genus, so we have shown all three definitions of genus are equal.