Basic Chat:

We will use elementary complex analytic methods to study complex projective curves. Basically we will take whatever road is easier, not worrying about being "pure".

Elliptic Functions: (Ref: Ahlfors Ch 7)

Fix \( \lambda_1, \lambda_2 \in \mathbb{C} \) that are linearly independent over \( \mathbb{R} \). \( \Lambda = \Lambda(\lambda_1, \lambda_2) = \mathbb{Z} \lambda_1 + \mathbb{Z} \lambda_2 \)

This gives us a lattice, a discrete subgroup of \( \mathbb{C} \).

Def: An elliptic function (wrt \( \Lambda \)) is a meromorphic function \( f(z) \) on \( \mathbb{C} \) which is periodic wrt to \( \Lambda \),

ie. \( f(z + \lambda) = f(z) \), \( \forall \lambda \in \Lambda \), \( z \in \mathbb{C} \).

Goal: Understand the existence and properties of elliptic functions.

Remark: \( \{ \text{elliptic functions} \} \subset \{ \text{meromorphic on } \mathbb{C}/\Lambda \} \)

Prop: The only analytic elliptic functions are constant.

Notation: Denote by \( \mathcal{P} \subset \mathbb{C} \) the closed region spanned by \( \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \).

\[ P_a = P + a \quad : \text{translate of } P \text{ by } a \text{ for } a \in \mathbb{C}. \]
Proof (Prop): Any \( f(z) \) is an entire \( \Lambda \)-periodic function.

If \( |f(z)| \) is bounded on \( \Lambda \) since \( \Lambda \) is compact. Now by
periodicity, it is everywhere bounded. Hence constant by
Jouwville's Thm.

Now assume \( f(z) \) meromorphic (always \( \Lambda \)-periodic).

Fix \( a \in \mathbb{C} \) s.t. \( f(z) \) has no zeroes or poles on \( \partial \Lambda_a \).

Thm (Residue Thm): The sum of the residues of \( f(z) \) inside \( \Lambda_a \)
is 0.

Proof: By the classical Residue Thm,

\[
\sum \text{res of } f(z) \text{ in } \Lambda_a = \frac{1}{2\pi i} \int_{\partial \Lambda_a} f(z) \, dz.
\]

By periodicity,

\[
\int_{\Lambda} = \int_{\Lambda'}, \quad \int_{\Pi} = \int_{\Pi'}, \quad \text{As we get}
\]

\[
\frac{1}{2\pi i} \int_{\partial \Lambda_a} f(z) \, dz = 0.
\]

Coroll: A nonzero constant elliptic function has the same number of
zeroes as zeroes (counting multiplicity) inside \( \Lambda_a \).

Proof: Consider the elliptic function \( \frac{f'(z)}{f(z)} \).

\[
\text{res}_{b} \left( \frac{f'(z)}{f(z)} \right) = \text{ord}_{b} (f(z))
\]

Use \( \sum \text{res} \left( \frac{f'}{f} \right) = 0 \), so \( \sum_{b \in \Lambda_a} \text{ord}_{b} (f) = 0 \).
Theorem ("First half of Ahlfors' Thm"): Let \( f(z) \) be a non-constant elliptic function. Let \( P_1, \ldots, P_r, q_1, \ldots, q_r \) be the zeroes and poles of \( f(z) \) inside \( P_a \), repeated according to their multiplicity. Then \( \sum p_i \equiv \sum q_i \pmod{\Delta} \).

Corollary: There is no elliptic function \( f(z) \) having a simple pole in \( P_a \).

Proof (Thm): By the classical Riemann Thm,
\[
\sum p_i - \sum q_i = \frac{-1}{2\pi i} \int_{\partial P_a} z \frac{f'(z)}{f(z)} \, dz
\]

As before,
\[
\int_{\partial P_a} z \frac{f'(z)}{f(z)} \, dz = \int_{\partial I} z \frac{f'(z)}{f(z)} \, dz + \sum_{\lambda \in \Lambda} z \frac{f'(z)}{f(z)} \int_{\partial I + \lambda z} z \frac{f'(z)}{f(z)} \, dz
\]

Recall \( \frac{1}{2\pi i} \int_{\partial I} z \frac{f'(z)}{f(z)} \, dz = \text{winding number of } f(\lambda) \text{ around the origin} \in \mathbb{Z} \).

\[
\int_{\partial P_a} z \frac{f'(z)}{f(z)} \, dz \in \lambda \mathbb{Z} \subseteq \Delta
\]

Similarly, for \( \int_{\partial P_a} z \frac{f'(z)}{f(z)} \, dz \). And so we get the result.
Proof (cont.): Note that if such a function exists, then there is exactly one zero as well. Since they are congruent, they must be equal. \( \therefore \)
Weierstrass $\wp$-form:

Define $\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda, \lambda \neq 0} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$

(*)

**Thm:** RHS of (*) converges uniformly on compact subset of $\mathbb{C} \setminus \Lambda$, to a meromorphic function having poles of order 2 and zero residues at each $\lambda \in \Lambda$, and no other singularities. $\wp(z)$ is even and elliptic w.r.t $\Lambda$.

**Proof:** For convergence, see Ahlfors Chapt 7.5.3.1.

The statement about the singularities and the statement that $\wp(z)$ is even are clear.

Need to show $\wp(z)$ is actually $\Lambda$-periodic.

Consider:

$$\wp'(z) = -\frac{3}{z^2} + \sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{2}{(z-\lambda)^3}$$

This is clearly periodic w.r.t $\Lambda$ because we can move summation etc.

Fix $\lambda \in \Lambda$.

$$(\wp(z + \lambda) - \wp(z))' = \wp'(z + \lambda) - \wp'(z) = 0.$$  

As for each $\lambda \in \Lambda$, there is $C = C_{\lambda}$ s.t.

$$\wp(z + \lambda) - \wp(z) = C_{\lambda}.$$  

Take $\lambda = \lambda_1$, $z = -\frac{\lambda_1}{3}$. So we get:

$$\wp\left( \frac{\lambda_1}{3} \right) - \wp\left( -\frac{\lambda_1}{3} \right) = C_{\lambda_1}.$$  

But $\wp\left( \frac{\lambda_1}{3} \right) - \wp\left( -\frac{\lambda_1}{3} \right) = 0$ because $\wp$ is even. Thus $C_{\lambda_1} = 0$.

Similarly, $\wp(z + \lambda_3) = \wp(z)$.
So our two "basic" elliptic functions are:

\[ g(\tau) = \frac{1}{\tau^2} + \sum_{\lambda \neq 0} \left( \frac{1}{(\tau - \lambda)^2} - \frac{1}{\lambda^2} \right) \]

\[ g'(\tau) = -2\sum_{\lambda} \frac{1}{(\tau - \lambda)^3}. \]

**Prop:** Laurent series for \( g \) is

\[ g(\tau) = \frac{1}{\tau^2} + \sum_{k \geq 1} (2k+1) G_{2k+2} \tau^{2k} \]

where \( G_{2m} = \sum_{\lambda \neq 0} \lambda^{-2m} \) (converge for \( m \geq 1 \)).

**Proof:** (sketch)

For \( m \geq 1 \), recall \( \frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + \cdots \).

\[ \frac{1}{(\tau - \lambda)^2} - \frac{1}{\lambda^2} = \lambda^{-2} \left( \left(1 - \frac{\tau}{\lambda} \right)^{-2} - 1 \right) \]

\[ = \lambda^{-2} \left( 2 \frac{\tau}{\lambda} + 3 \left( \frac{\tau}{\lambda} \right)^2 \cdots \right) \quad \text{for } |\tau| < |\lambda| \]

Now plug into the formula for \( g(\tau) \) and rearrange.

\[ \Box \]

**Thm:** (assert) \( g \) and \( g' \) generate the field of all elliptic functions

wrt \( \Lambda \), i.e.,

\[ \left\{ \text{elliptic forms} \right\} = \mathcal{O}(g, g') \]

"field"

**Next:** We want to show \( g, g' \) satisfy some polynomial equation,

i.e., \( g \) satisfies some differential equation.

**Def:** Given \( k \geq 0 \), define

\[ V_k = \left\{ \text{all elliptic forms } | \text{f has poles of order } \leq k \text{ on } \Lambda, \text{analytic on } \mathbb{C} \setminus \Lambda \right\} \]
$V_k$ is a $C$-vector space.

$V_0 = C$

$V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$

Claim 1: $\dim V_k - \dim V_{k-1} \geq 1$

Proof: Let $K > 2$, then we can write $K = 3m + 3n$ some $m, n \geq 0$.

Then $g^m(p')^n \in V_k \setminus V_{k-1}$.

Claim 2: $\dim V_k - \dim V_{k-1} \leq 1$

Proof: Take $f, g \in V_k \setminus V_{k-1}$. So we need to show $f, g$ are linearly dependent (not $V_{k-1}$). By the Laurent series are:

$f = a_0 + \cdots$

$g = b_0 + \cdots$

Then $bf - ag$ has a pole of order at most $K - 1$, i.e.,

$bf - ag \in V_{k-1}$.

Now we write down elements of these vector spaces:

$V_0 = V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq V_5 \subseteq V_6$

$1 \quad g \quad g' \quad g'' \quad g''' \quad g'''' \quad g^{\text{triple}}$

As we have 7 elements in $V_6$, which has dim 6. A there is some $C$-linear combination that is zero.
Corl: The 7 indicated functions are linearly dependent over \( \mathbb{C} \).

ie, \( \exists \) poly relation:

\[
(g^3)'^2 = a g^3 + b g^3 g = \cdots + e g + f
\]

where \( a, \ldots, f \in \mathbb{C} \).

Thus, there is some differential equation that \( g \) satisfies.

Thm: \( \ln \)身份,

\[
(g^3)'^2 = 4 g^3 + 6 g^3 g + 14 g^3.
\]

Let \( g_2 = 6 g \), \( g_3 = 14 g \), then we have:

\[
(g^3)'^2 = 4 g^3 + g_2 g + g_3.
\]

Remark: Formally, \( w = g(z) \), then

\[
\frac{d w}{d z} = \sqrt{4 w^3 + g_2 w + g_3}.
\]

So, \( \frac{d w}{\sqrt{4 w^3 + g_2 w + g_3}} = d z \).

ie, \( z = \int \frac{d w}{\sqrt{4 w^3 + g_2 w + g_3}} \).

This is an elliptic integral.

ie,

\[
Z - Z_0 = \int \frac{g(z)'}{\sqrt{4 g^3 + g_2 g + g_3}}
\]

Can see Ahlfors pg 368, chap 5 & 6 to see the details.
**Complex Manifolds:**

**Def (informal):** A complex manifold of complex dimension $m$ is a 2nd countable Hausdorff space $X$, with a covering by open sets $\{ U_\alpha \}$ together with homeomorphisms $c_{\alpha\beta} : U_\alpha \rightarrow V_\alpha \subseteq \mathbb{C}^m$

![Diagram of complex manifolds]

Set the transition functions $g_{\alpha\beta} : c_{\alpha}(U \cap U_\beta) \rightarrow c_{\beta}(U \cap U_\alpha)$ where $g_{\alpha\beta} = c_{\beta} \circ c_{\alpha}^{-1}$ are analytic.

**Example:** $\mathbb{P}^1 = \text{Riemann sphere} = \{ C \cup \{ \infty \} \}$

![Diagram of Riemann sphere]

$\mathbb{P}^1 \setminus \{ \infty \} = \mathbb{C}$

$\omega = \frac{1}{z}$

$\mathbb{P}^1 \setminus \{ 0 \} = \mathbb{C}$

So $g_{1,0}(z) = \frac{1}{\omega}$.

This definition is informal because we should be defining an equivalence relation on the set of coverings of $X$, etc.
Def: A Riemann surface is a complex manifold of complex dimension 1.

Example: \( \mathbb{P}^n \) : homogeneous coordinates \([T_0 : \cdots : T_n]\)

Chart: \( U_i = \{ T_i \neq 0 \} \ni [T_0, \cdots, T_n] = \left[ \frac{T_0}{T_i}, \ldots, 1, \frac{T_n}{T_i} \right] \)

Translation: \( g_{10} : V_0 \to V_1 \) (quotations are because it is not defined in all of \( V_0 \) given the overlap)

\[
\begin{align*}
U_0 & : [z_1, z_2, z_3] = [\frac{1}{z_1}, 1, \frac{z_2}{z_1}, \frac{z_3}{z_1}] \in U_1 \\
g_{10} & : (z_1, z_2, z_3) \to (\frac{1}{z_1}, \frac{z_2}{z_1}, \frac{z_3}{z_1}) \in V_1
\end{align*}
\]

Example: \( \Lambda \subseteq \mathbb{C} \) lattice

\( \begin{array}{c}
\Lambda \subseteq \mathbb{C} \\
= \mathbb{C}/\Lambda \\
\approx S' \times S'
\end{array} \)

Let \( V(a) \) be a connected nbhd of \( a \in \mathbb{C} \) sufficiently small so that two points of \( V(a) \) are congruent modulo \( \Lambda \).

\( \pi: V(a) \to U(a) = \pi(V(a)) \)

So \( \phi^a = (\pi|U(a))^{-1}: V(a) \to U(a) \) is a local coordinate.
The transition functions
\[ g_{w_1} : \mathbb{V}(a) \rightarrow \mathbb{V}(b) \]
are
\[ \mathbb{Z} \rightarrow \mathbb{Z} + (b-a) \]

Example: (This takes some thought)

A nonsingular, connected, quasi-projective complex variety is a complex manifold. (This is met the Zariski topography, it gets its topology from \( \mathbb{C} \mathbb{P}^n \).

\[ \mathbb{Z} = \{ (z,w) \mid f(z,w)=0 \} \leq \mathbb{C}^2 \] is a nonsingular plane curve.

\[ V \ni \mathbb{Z}, \quad \frac{\partial f}{\partial z} (p) \neq 0 \text{ or } \frac{\partial f}{\partial w} (p) \neq 0 \]

\[ \text{If } \frac{\partial f}{\partial w} (p) \neq 0, \text{ then we can use } z \text{ as a local coordinate.} \]

\[ \text{If } \frac{\partial f}{\partial z} (p) \neq 0, \text{ then use } w \text{ as a local coordinate.} \]

Transition functions: implicit function theorem is true for analytic functions. If \( \frac{\partial f}{\partial w} (0) \neq 0 \), then there is an analytic function \( \phi(z) \) s.t. \( \mathbb{Z} \) is locally the graph of \( w = \phi(z) \), i.e., \( \mathbb{Z} \cap \ker \phi = \{(z, \phi(z)) \mid z \in \mathbb{Z} \} \).
Pictorial Example: "The Riemann surface of an analytic function"

\[ w^3 - z^2(z-1) = 0 \]

"\[ w = \pm \sqrt[3]{z^2(z-1)} \]"

\[ \mathbb{X} \subset \mathbb{W} \]
\[ \downarrow \]
\[ \mathbb{C} \rightarrow \mathbb{Z} \]

This picture is not the Riemann surface because it is singular.

: Pulling points apart.
More formal definition:

\[ X = \text{Hausdorff space.} \]

**Def:** Two charts \( \varphi : U \to V, \varphi' : U' \to V' \) are compatible if 
\[ \varphi' \circ \varphi : \varphi(U \cap U') \to \varphi'(U \cap U') \] is analytic. (on \( U \cap U' \)).

**Def:** An atlas on \( X \) is a collection \( \mathcal{A} = \{ \varphi \colon U \to V \} \) of compatible charts whose domains cover \( X \).

**Def:** Two atlases \( \mathcal{A}, \mathcal{B} \) are equivalent if every chart of \( \mathcal{A} \) is compatible with every chart of \( \mathcal{B} \).

**Lemma:** Every atlas is contained in a unique maximal atlas.

**Def:** A complex manifold is a 2nd countable Hausdorff space together with an equivalence class of atlases. (equivalently, a max. atlas).

**General Philosophy:**

Any "intrinsically defined" analytic notion makes sense on a complex manifold.

**Example:** \( X \) = complex manifold.

A function \( f : X \to \mathbb{C} \) is holomorphic if \( f \circ \varphi_i^{-1} \) is holomorphic for all charts \( \varphi_i : U_i \to V_i \).

\[ X \supseteq U_i \xrightarrow{f} \mathbb{C} \]

\[ \varphi_i : \mathbb{C} \to V_i \xrightarrow{f \circ \varphi_i^{-1}} \]

Note that this does not depend on the choice of the chart in the sense that:
\[ U_i \subseteq \mathbb{C} \rightarrow U_j \]
\[ \varphi_i \downarrow \quad \downarrow \varphi_j \]
\[ V_i \xrightarrow{g_{ji}} V_j \]
\[ f \circ \varphi_i^{-1} = f \circ \varphi_j^{-1} \circ g_{ji}. \]

Each in holomorphic if the other is. \[ g_{ji} \] is biholomorphic.

Similarly for meromorphic functions.

**Example:** \( (\text{mero funs}) \) \( \rightarrow \) \( (\text{elliptic funs}) \)

**Notation:** \( \mathcal{C}(x) = \text{field of meromorphic functions on } \mathbb{C} \).

**Def.:** A **continuous map** \( f: \mathbb{C} \rightarrow \mathbb{C} \) between complex manifolds \( \mathbb{C}, \mathbb{C} \) is **holomorphic** if it is given in local coordinates by analytic functions, i.e.,
after refinement, look for chart

\[ \mathbb{C} \ni U_i \xrightarrow{\varphi_i} V_i \subseteq \mathbb{C} \]
\[ \mathbb{C} \ni W_j \xrightarrow{\psi_j} O_j \]

s.t. \( f(U_i) \subseteq W_{ji}; \psi \).

\[ \begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C} \\
U_i & \xrightarrow{\varphi_i} & W_{ji} \\
\varphi_i \downarrow & & \psi_j \\
V_i & \xrightarrow{\psi_j \circ \varphi_i^{-1}} & V_j \end{array} \]

, \( \psi_j \circ \varphi_i^{-1} \) is holomorphic.
Example: \( \pi : \mathbb{C} \to \mathbb{C}^n \) is analytic.

Example: Fix \( \Lambda \subset \mathbb{C} \) lattice, \( \mathbb{X} = \mathbb{X}_{\Lambda} = \mathbb{C}/\Lambda \).

Define \( \tilde{\varphi} : \mathbb{X} \to \mathbb{P}^2 \) via \( \tilde{\varphi}(z) = [y(z), y'(z), 1] \).

Proof: \((*)\) defines a holomorphic map \( \tilde{\varphi} \) whose image is contained in the algebraic curve \( E = E_{\Lambda} = \{ z \in \mathbb{C}^3 : y^3 - 4x^3 - 9x - g_3 = 0 \} \) if
\[
\left\{ y^3, -14y^2x, -93y, 92x^2, -9, 2 \right\} = 0.
\]

Proof: This just comes down to understanding what \((*)\) means.

First consider
\[
\begin{array}{c}
\mathbb{C} \\
\downarrow
\end{array} \xrightarrow{\varphi} \mathbb{P}^2
\]
\[
\mathbb{X} = \mathbb{C}/\Lambda \xrightarrow{\tilde{\varphi}} \mathbb{P}^2
\]
\[
\tilde{\varphi}(z) = [y(z), y'(z), 1]
\]

\( \tilde{\varphi} \) is analytic map \( \mathbb{C}/\Lambda \to \mathbb{C}' = \mathbb{C} \setminus \{ 0 \} \subset \mathbb{P}^2 \)
\[
\pi \longmapsto (y(z), y'(z)).
\]

This defines an analytic map
\[
\mathbb{X} \setminus 0 \longrightarrow \mathbb{C} \subset \mathbb{P}^2
\]
\[
\mathbb{C} \setminus \pi(\Lambda)
\]

Note: For \( z \neq 0 \), \([y(z), y'(z), 1], [z^3y(z), z^2y'(z), z^3] \)
\[
\lim_{z \to 0} [z^3y(z), z^2y'(z), z^3] = [0, -2, 0] = [0, 1, 0] \text{ in } \mathbb{P}^2.
\]

For \( z \) near \( 0 \), \( \tilde{\varphi}(z) = \left( \frac{y(z)}{y'(z)} \right) \subset \mathbb{C}' = \mathbb{C} \setminus \{ 0 \} \subset \mathbb{P}^2 \).

This is analytic for \( z \) near \( 0 \) (in any \( \Lambda \)), so define
\[
\tilde{\varphi} : \mathbb{C} \to \mathbb{P}^2
\]
hence $\phi : \mathbb{X} \rightarrow \mathbb{P}^3$

Now, the fact the image lies in the curve is just the differential equation we already derived. 

*General Principle:*

$\mathbb{X} = \{s, \ldots, s \in C(\mathbb{X}) \text{ mon-const. mono func.}\}$

Then $\phi : \mathbb{X} \rightarrow \mathbb{P}^r$

$x \mapsto [1, f_1(x), \ldots, f_r(x)]$

defines a holomorphic map. Moreover, any nonzero holomorphic map like this.

*Warning:* If $\mathbb{X}$ is a complex manifold of dim 2, then the corresponding statement can fail.

$f_1 = \frac{y}{x}$ on $C^2$

$[1, \frac{y}{x} = [x, y] \text{ does not define an analytic map from } C^2 \text{ to } \mathbb{P}^1.$

This is because of the origin. Do it at a zero, pole, something else?

For manifolds of dim 1 the zeroes and poles are isolated so this problem doesn't come up.

*Thm:* Let $\Lambda \subseteq C$ be a lattice, $\mathbb{X} = C/\Lambda$

$g_0, g_1 \in C \text{ constants determined by } \Lambda.$

1. Curve $E = E_\lambda = \{y^2 - 4x^3 - g_2x - g_3 = 0\}$
   is nonsingular, compact, and hence a R.S.

2. Mapping $\phi : \mathbb{X} \rightarrow E \ni ([x(2), y(2), t])$ defines an isomorphism of complex manifolds.
Recall:
\[ \Lambda \in \mathbb{C}, \, \mathbb{E} = \mathbb{C}/\Lambda \]
\[ g_2, g_3 \in \mathbb{C} \quad \text{constants associated to } \Lambda \]
\[ E = E^\Lambda \subseteq \mathbb{P}^3 \quad \ldots \]
\[ = \text{cyclic } \, y^3 = 4x^3 + g_2x + g_3 \, \in \mathbb{C}^3 \]

**Thm:**
1. \( E \) is nonsingular
2. Map
\[ \phi : \mathbb{C} \rightarrow E \subseteq \mathbb{P}^3 \quad \text{is an isomorphism.} \]

**Proof:** One can check that \( y^3 = 4x^3 + g_2x + g_3 \) is nonsingular in all \( \mathbb{C} \setminus \mathbb{P}^3 \) iff poly \( 4x^3 + g_2x + g_3 \) has distinct roots.

Let \( \lambda_1, \lambda_2, \lambda_3 \in \Lambda \) be a basis. The three zeroes of \( \phi' \) are
\[ \frac{\lambda_1}{2}, \quad \frac{\lambda_2}{2}, \quad \frac{\lambda_3}{2} \]

(in one period parallelogram) This is because \( \phi' \) is odd and so we have:
\[ \phi'(-\frac{\lambda_1}{2}) = \phi'\left(-\frac{\lambda_2}{2}\right) \]
\[ = \phi'\left(-\frac{\lambda_3}{2}\right) \]
\[ = 0 \]

The roots of \( 4x^3 + g_2x + g_3 \) are then \( \phi\left(\frac{\lambda_1}{2}\right), \, \phi\left(\frac{\lambda_2}{2}\right), \, \phi\left(\frac{\lambda_3}{2}\right) \).

So we need to show these are all distinct.

Consider \( \phi(2) - \phi\left(\frac{2}{3}\right) \). This vanishes at \( \frac{2}{3} \), even and periodic, so it has a double zero at \( \frac{2}{3} \). So it can't have any other zeros.
in a period parallelogram (only $h=1$), it follows that
\[
\frac{A_2}{\Delta} \quad \text{and} \quad \frac{A_1 \times A_2}{\Delta} \quad \text{can't be roots of} \quad \phi(z) = \phi\left(\frac{A_1}{\Delta}\right).
\]
As the three values are distinct, so $E$ is nonsingular.

(2) **Claim**: $\phi : \mathbb{R} \to E$ is 1-1 and onto.

**Proof** (surj):

![Diagram](image)

Take $(x, y) \in E$. Consider $\phi(z) - x$. This is a constant
elliptic function, so it has zeros at $z = a, -a$ for some $a$. Thus $\phi(a) = x$. So then

\[
(\phi'(a))^2 = 4x^3 + 9x + g_3 = y^2.
\]

Thus, either $y = \phi'(a)$ or $y = \phi'(-a)$.

$n$: Exercise.

To complete the proof, we use the following lemma.

**Lemma**: A 1-1, surj. holomorphic map $f : \mathbb{C} \to \mathbb{C}$ between
Riemann surfaces is an isomorphism.

**Sketch of Proof**: Need to show $f^{-1}$ is holomorphic. Let local coordinates,

\[
\begin{align*}
\mathbb{C} & \quad \text{local coord.} \\
\mathbb{C} & \quad \text{local coord.} \\
\varphi(z) & = f(0) = 0
\end{align*}
\]

\[
\mathbb{C} = f(z) = z^e \quad e > 1 \quad (\text{locally!})
\]
$f(z)$ maps $\mathbb{C}$ to $\mathbb{C}$ locally. Now just take $e^z$ of the extra stuff.

Now since $f$ is 1-1 implies $e^1 = f^{-1}$ is analytic. \[\Box\]

**Caution:** This lemma is NOT true if the curve is 'singular'. Think of $\mathbb{A}^1$ and a curve by a cauop.

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**Group Law on $E$:**

One can define a law on $E$ as indicated:

![Diagram of group law on $E$]

- $[0,1,0]$ - pt at infinity.
- $[1,0,0]$ - origin.
- $P, Q, R, P+Q, R-P.Q$.
- Vertical line passes through $P$ at $\infty$.

**Assuming fact:** This addition makes $E$ into an additive group.

**Prop:** This realizes the natural group structure on $X = \mathbb{C}/\Lambda$

by $X \cong E$.

This basically breaks down to the addition formula for $f(x)$. 
If $E$ is defined over $\mathbb{Q}$, then we get a group $E(\mathbb{Q})$.

Consider the curve $E: y^2 = 4x^3 + 9x + 9$, with $RHS$ having distinct roots. Is $E$ of the form $E/\Lambda$ for some $\Lambda$? The answer is yes, but it isn't necessarily obvious how we get $\Lambda$.

**Meromorphic Functions on $E/\Lambda$:**

Let $C(x) = \text{field of meromorphic functions on } E$

- $C(x) = C(g, g')$
- ($C(E) = \text{rational functions on } E$)

(a) $0 \neq f \in C(x)$ has the same $\#$ of zeroes and poles.

Fix $p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{R}$, allowing repetitions.

**Theorem (Abel):** There exists a meromorphic function $f \in C(x)$

- pole at $p_i$ and zero at $q_i$ iff $\sum p_i = \sum q_i$
- in $E/\Lambda$.

**Example:**

- $E$
- $P + Q + R = 0$ iff colinear.

Any $P, Q, R$ are colinear. Then we should be able to find
If we take $P, Q, R$ as zeroes and a triple pole at $\infty$.

$L(x, y)$ is the line that $P, Q, R$ lie on.

Can take this line as the numerator and line at $\infty$ as the denominator.

We'll introduce a new function in $\mathbb{C}$, "theta function": Given $\Lambda \subseteq \mathbb{C}$,

$$\sigma(z, \Lambda) = \sigma(z)$$

$$= \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right)e^{\frac{z}{\lambda} + \frac{\overline{z}}{\lambda}}$$

**Prop:**

(a) $\sigma$ converges to an entire function with simple zeroes at lattice pts.

(b) Next time.

(c) For $\lambda \in \Lambda$, $\exists$ $a = a_\lambda$, $b = b_\lambda$ s.t.

$$\sigma(z + \lambda) = e^{az + b} \sigma(z)$$

We will want to look at ratios of the form:

$$\frac{\prod \sigma(z - q_i)}{\prod \sigma(z - p_i)}$$
Theorem (Azumi): Given \( p_1, p_2, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{C}/\Lambda \). Then there exists a meromorphic form \( f \) on \( \mathbb{C} \) with poles at \( p_i \) and zeroes at \( q_i \) iff \( \sum p_i = \sum q_i \) in \( \Lambda \).

Given \( \Lambda \subseteq \mathbb{C} \), define \( \sigma(z) = \prod_{\lambda \in \Lambda} \left( 1 - \frac{z}{\lambda} \right) e^{\left( \frac{z^2}{2} \frac{1}{(\lambda)^2} \right)} \).

Proof: (a) Infinite part converges to define entire form \( \sigma(z) \) w/ simple zeroes on \( \Lambda \), modulo Eisen.

(b) \( \frac{d^2}{dz^2} \log \sigma = -\Phi(z) \)

(c) Given \( \lambda \in \Lambda \), \( \exists \ a = a_\lambda, b = b_\lambda \ s.t.

\[ \sigma(z + \lambda) = e^{(az + b)} \sigma(z) \]

Proof: (a) see Azumi

(b) \( \log \sigma = \log z + \sum_{\lambda \neq 0} \left[ \log \left( 1 - \frac{z}{\lambda} \right) + \left( \frac{z^2}{2} \frac{1}{(\lambda)^2} \right) \right] \)

Differentiate twice term by term:

\[ \frac{d^2}{dz^2} \log \sigma = -\frac{1}{z^2} + \sum_{\lambda \neq 0} \left[ -\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2} \right] \]

\[ = -\Phi(z) \]

(c) Fix \( \lambda \in \Lambda \). By (b),

\[ \frac{d^2}{dz^2} \left( \log \sigma(z + \lambda) - \log \sigma(z) \right) = 0 \quad (\Phi(z + \lambda) = \Phi(z)) \]

And \( \log \sigma(z + \lambda) = \log \sigma(z) + (az + b) \quad (\text{do not edit this is } 0) \)

\[ \sigma(z + \lambda) = e^{az + b} \sigma(z) \]
Proof (Wld's Thm): We have already shown $^\exists$.

Fix points $c_1, \ldots, c_n \in \mathbb{C}$, $n_1, \ldots, n_k \in \mathbb{Z}$, look for an elliptic function $f(z)$ s.t. $\text{ord}_c f(z) = n_c$ (no other zeros or poles).

Assume $\sum n_i = 0$, $\sum n_i c_i \equiv 0 \pmod{\Lambda}$. We have just translated the statement into multiplicities.

We can assume wlog that $\sum n_i c_i = 0$.

Consider
\[
f(z) = \prod_{i=1}^{n} (z - c_i)^{n_i}.
\]

This visibly has the right number of zeros and poles.

Fix $\lambda \in \Lambda$. We need $f(z)\lambda$ to be periodic.

\[
f(z + \lambda) = \prod_{i=1}^{n} (z + \lambda - c_i)^{n_i}.
\]

\[
= \prod_{i=1}^{n} \left[ \sigma(z - c_i) e^{aq(z-c_i)+b} \right]^{n_i}.
\]

\[
= \prod_{i=1}^{n} f(z) \left[ e^{az+b} \cdot e^{aq(c_i)} \right]^{n_i}.
\]

\[
f(z) \sigma(z) (a(z-c_i) + b) \left( \sum n_i \right).
\]

\[
f(z) \sigma \left( \sum n_i (z-c_i) \right) \left( \sum n_i = 0 \right).
\]

\[
f(z) \left( \sum n_i c_i = 0 \right).
\]

\[
\forall \lambda_1, \lambda_2 \in \mathbb{C}, \text{ when is } \mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2? \text{ We will come back to this later on in the course.}
\]
Riemann Surfaces:

Let \( X \) = compact, connected R.S.

As \( X \) is a closed, oriented \( 2 \)-manifold. (oriented because hole retired present orientation)

Theorem: (Classification of surfaces): \( X \) is diffeomorphic to a sphere \( S^2 \) with \( g \) handles attached.

\[
\begin{align*}
& g=0 &  & g=1 \\
& g=2 &  & g=3 \\
& \vdots &  & \vdots
\end{align*}
\]

\( g \) is called the genus of \( X \) and determines \( X \) up to diffeomorphism.

Remark: It is not true that all Riemann surfaces of given genus \( g \) are isomorphic as Riemann surfaces.

However, genus is still the critical invariant of a R.S.

Prop: \( X \) compact, connected, then any homomorphic fcn on \( X \) is constant.
Proof: Day \( f : \mathbb{R} \to \mathbb{C} \) is holomorphic. Compactness implies
\[ f(x) \] attains a max at some \( p \in \mathbb{R} \). Look locally
near \( p \). By the maximum modulus principle given f locally constant near \( p \). Now by connectedness and
\#3 in Huse 2 implies \( f \) is globally constant.

Example: Consider the function \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \), where we
view \( \mathbb{P}^1 \) as a curve from \( \mathbb{P}^1 \). How do we compute it
residue at the origin?

\[
\text{Res}_0 \left( \frac{1}{z} \right) = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z} = -\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z}
\]

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

Where is the mistake?

*Functions cannot be integrated on a manifold!*

*Meromorphic forms on a R.S. don’t have residues!*

\[
\begin{bmatrix}
\int_{c} \frac{dz}{z} = \int_{-c} -\frac{dw}{w} \\
\int_{-c} \frac{dw}{w} = \int_{c} -\frac{dz}{z}
\end{bmatrix}
\]

So everything is ok.

As if we want to integrate, we need to use the change of
variables formula, i.e., differential forms.

The things you can integrate on a R.S. are 1-forms.

Similarly, meromorphic 1-forms are what have residues. (can integrate around a circle containing)
Def: A \textit{holomorphic 1-form} on a R.S. \( \mathbb{X} \) is \( C^\infty \)-valued 1-form which in local coordinates can be expressed as \( \omega = f(z) dz \) locally.

\[ dz = dx + i dy, \quad f(z) \text{ analytic}. \]

A \textit{meromorphic 1-form} is defined similarly except \( f \) is allowed to be meromorphic.

Note: Check the definition doesn't depend on coordinates:

\[ \omega = g(w) \]

\[ d\omega = g'(w) dw \]

\[ f(z) d\omega = f(z(w)) \cdot g'(w) dw \quad (\text{the LHS holds (meas) iff RHS does}). \]

This also shows how they change coordinates.

Examples: 1 on \( \mathbb{P}^1 \)

\[ \frac{dz}{z} \] is a \textit{meromorphic} 1-form.

( in words, \( w = \frac{1}{z} \), centered at \( \infty \), \( \frac{dz}{z} = -\frac{dw}{w} \)).

2 on \( \mathbb{X} = \mathbb{C}/\Lambda \), can write \( d\omega \) so \textit{holo} 1-form.

\[ \begin{cases} \mathbb{C} \rightarrow \mathbb{X} = \mathbb{C}/\Lambda, & \text{1 holo 1-form } \omega \text{ on } \mathbb{X} \\ \text{s.t. } \pi^* \omega = d\omega & \end{cases} \]

The transition in this case are \( \omega = e^{i\theta} \), so \( d\omega = d\theta \).

As on \( \mathbb{C}/\Lambda \), there is a natural identification:

\[ \{ \text{holo 1-forms} \} \leftrightarrow \{ \text{forms} \} \]
Def: If $w$ is a meromorphic 1-form on R.S. $\mathcal{X}$, $w = f(z)dz$ locally, then $ord_p(w) = ord_p(f(z))$.

Note: This is well-defined because when we change coordinates, $g'(w) \neq 0$ for all $w$.

Example: If $f$ is a meromorphic form on $\mathcal{X}$, $df$ is a meromorphic 1-form:

locally, $df = f'(z)dz$.

If $X \in \mathcal{X}$ is a (reasonable) path, and $w$ is a holomorphic 1-form, then

\[ \int_X w \text{ is defined.} \]

The idea is that the change of basis formula is built into these things, so we can integrate locally w/o changing answers.

Prop: A holomorphic 1-form $w$ on R.S. $\mathcal{X}$ is closed, i.e., $dw = 0$.

Proof: Write $z = x + iy$, $w = f(z)dz$ locally. So we have

\[ w = (u + iv)(dx + idy) \]

\[ = (udx - vdy) + i(vdx +udy) \]

\[ dw = (-\frac{2v}{\partial y} - \frac{2x}{\partial x})dx \wedge dy + i(-\frac{2u}{\partial y} + \frac{2v}{\partial x})dx \wedge dy \]

\[ = 0 \text{ by Cauchy-Riemann equations.} \]

Warning: Can define holomorphic forms on complex manifolds of dim $\geq 1$, but their holomorphic 1-forms need not be closed.

\[ w = z dz \text{ on } \mathbb{C}^2 \text{ when coordinates on } \mathbb{C}^2 \text{ are } (z, w). \]
On a smooth projective variety, holomorphic forms are closed for Hodge theoretic global reasons.

Remark: Recall that closed forms determine (delHam) cohomology classes in $H^1(X, \mathbb{C})$.

What are cohomology classes determined by holomorphic \[ \text{oriented loop around p.} \]

This is independent of $\gamma$ since $\omega$ is closed. (Stokes Thm).

Thm (Residue Thm): Let $\omega$ be a non-zero one-form on a compact R.S. $X$. Then \[ \sum_p \text{Res}_p(\omega) = 0. \]

(Res$\omega$=0 if $\omega$ is holomorphic).

Proof:

Choose a small open disk around each pole as shown.

Let $M = X \setminus \text{(univ. of disks)}$.

Then $\partial M = \text{univ. of circles surrounding the poles}$.

So \[ \sum \text{Res}_p(\omega) = \pm \frac{1}{2\pi i} \int_{\partial M} \omega = \pm \frac{1}{2\pi i} \int_M d\omega = 0. \]
Corl: Let $f$ be a non-zero meromorphic function on a compact R.S. $X$. Then $f$ has the same number of zeroes and poles (counting multiplicity).

\[ \sum_{p \in X} \text{ord}_p(f) = 0. \]

\[ \text{Pf:} \] Consider the meromorphic $1$-form $\omega = \frac{df}{f}$. Local calculation as before shows that $\text{Res}_p(\omega) = \text{ord}_p(f)$. Now apply the residue thm.

\[ \text{Thm:} \] Let $\omega, \eta$ be non-zero meromorphic $1$-forms on compact R.S. $X$.

Then \[ \sum_{p \in X} \text{ord}_p(\omega) = \sum_{p \in X} \text{ord}_p(\eta) \]

i.e., $\text{(no. of zeroes - no. of poles)}$ is the same for any two meromorphic 1-forms.

\[ \text{Proof:} \] The claim is that we can view $f = \frac{\omega}{\eta}$ as a meromorphic function $X$.

\[ \text{Meaning of} \ \frac{\omega}{\eta} : \text{Locally write} \ \omega = \phi(z)dz, \ \eta = \psi(z)dz. \]

\[ \frac{\omega}{\eta} = \frac{\phi(z)}{\psi(z)} \text{ locally.} \]

\[ \text{Change coordinates:} \quad z = g(w) \quad dz = g'(w)dw \]

\[ \frac{\omega}{\eta} = \frac{g'(w)}{\psi(g(w))} \frac{\psi'(g(w))g'(w)dw}{\eta(g(w))} = \frac{\phi(g(w))dw}{\psi(g(w))} = \frac{\phi(g(w))}{\psi(g(w))} \]

So this is a global meromorphic function.

\[ \text{ord}_p\left( \frac{\omega}{\eta} \right) = \text{ord}_p(\omega) - \text{ord}_p(\eta). \]

By previous corl, $\sum \text{ord}_p\left( \frac{\omega}{\eta} \right) = 0$. \[ \square \]
Examples: 1. Let $\omega$ be any mero 1-form on $\mathbb{P}^1$. Then

$$\sum_{\text{ord}(\omega)} = -2.$$

So there is no non-zero hole forms on $\mathbb{P}^1$.

PF: Evaluate for $\frac{dz}{z}$, $-\frac{dz}{z}$.

2. Let $\omega$ be any mero 1-form on $\mathbb{X} = \mathbb{C}/\Lambda$. Then

$$\sum_{\text{ord}(\omega)} = 0$$

Consequently, any hole from on $\mathbb{X}$ is $\omega = (\text{const})\, dz$.

PF: Evaluate for $\omega = dz$.

If $\omega$ is hole 1-form on $\mathbb{X} = \mathbb{C}/\Lambda$, then

$$\frac{\omega}{dz}$$

is a mero form up one pole, so constant. \[ \Box \]
Riemann-Hurwitz and Applications:

\( \mathbb{X}, \mathbb{Y} \) compact R.S.

\[ \mathbb{X} \rightarrow \mathbb{Y} \quad \text{holomorphic map.} \]

Riemann-Hurwitz relation \( g(\mathbb{X}) + g(\mathbb{Y}) \) (genus).

Recall:

\[ \epsilon(\mathbb{X}) = 2g - \delta \quad \text{# vertices - # edges + # \delta's}. \]

Recall: Local normal form

\[ F: \mathbb{X} \rightarrow \mathbb{Y} \quad \text{monic}. \quad \text{Local map deg at } p \in \mathbb{X} = \exists ! m \geq 1 \text{ s.t.} \]

\[ \forall \text{ chart } \varphi_2: U_2 \rightarrow V_2 \text{ centered at } F(p_1), \forall \text{ chart } \varphi_1: U_1 \rightarrow V_1 \]

centered at \( p \) s.t. \( \varphi_2(F\varphi_1^{-1}(z)) = z^m \)

**Def.**

\[ m = \text{mult}_p F = \text{mult}_p \varphi_1 F \circ \varphi_2^{-1} \]

\[ \text{deg } F = \sum_{p \in F(\mathbb{Y})} \text{mult}_p F \quad \text{for } \mathbb{X} \rightarrow \mathbb{Y}. \]

\( \text{If } \text{mult}_p F > 1, \text{ we call } p \text{ a ramification point.} \)

\( \text{If } y \in \mathbb{Y} \text{ is the image of a ramification point, then } y \)

\( \text{is called a branch point.} \)

**Example:**

\[ \begin{array}{c}
\text{Note: previous or branch pt}
\text{isn't necessarily a ram.}
\text{pt!}
\end{array} \]
Thm: (Riemann-Hurwitz): \( F : \mathbb{X} \rightarrow \mathbb{Y} \) monodromy holomorphic map between compact R.S. Then

\[
2g(\mathbb{X}) - 2 = (2g(\mathbb{Y}) - 2) \deg F + \sum_{p \in \mathbb{X}} [\text{mult}_p F - 1].
\]

Proof: Triangulate \( \mathbb{Y} \) s.t. every branch point is a vertex:

\[
\begin{align*}
\text{Ex:} & \quad d = 3 \\
& \quad \begin{array}{c}
\quad \text{P} \text{ 2\rightarrow 2} \\
\quad \text{X} \\
\quad \text{b}
\end{array}
\end{align*}
\]

Let this triangulation to \( \mathbb{X} \).

\( V_\mathbb{Y} = \# \text{vertices in} \ \mathbb{Y} \quad V_\mathbb{X} = \# \text{vertices in} \ \mathbb{X} \)

\( E_\mathbb{Y} = \# \text{edges in} \ \mathbb{Y} \quad E_\mathbb{X} = \# \text{edges in} \ \mathbb{X} \quad \text{etc.} \)

- Each triangle lifts to precisely \( \deg F \) triangles.

\( t_\mathbb{X} = \deg F t_\mathbb{Y} \)

- Similarly, \( e_\mathbb{X} = \deg F e_\mathbb{Y} \).

The tricky thing is keeping track of the number of vertices.

Take \( q \in \mathbb{Y} \) vertex.

\[
|F'(q)| = \sum_{p \in F(q)} 1 + \deg F - \sum_{p \in F(q)} [\text{mult}_p F]
\]

\[
= \deg F + \sum_{p \in F(q)} [1 - \text{mult}_p F]
\]

\[
V_\mathbb{X} = \sum_{q \in \mathbb{Y}} \left( \deg F + \sum_{p \in F(q)} [1 - \text{mult}_p F] \right)
\]

\[
= V_\mathbb{Y} \deg F + \sum_{p \in \mathbb{X}} [1 - \text{mult}_p F]
\]
\[ 2g(z) - 2 = -e(z) = -v_x + e_x = l_x \]

\[ = -\deg F v_x + \sum_{p \in \mathbb{P}} [\text{mult}_p - 1] + \deg F e_x - \deg F v_y \]

\[ = \deg F (\partial g(y) - 2) + \sum_{p \in \mathbb{P}} [\text{mult}_p - 1] \]

**Applications:**

**Hyperelliptic surfaces:**

\[ \Sigma : y^2 = h(x) \subseteq \mathbb{C}^2 \]

\[ h(x) \text{ poly, } \deg h(x) = 2g+1. \]

\[ h(x) \text{ has distinct roots (guarantees non-singularity).} \]

We need to compactify \( \Sigma \) to be able to apply R-H.

\[ U = \{ (x,y) \in \Sigma \mid x \neq 0 \} \]

\[ K(\Sigma) = z^{2g+2} h(\frac{1}{z}). \]

\[ \Sigma^0 = U \xrightarrow{\phi} \mathbb{C}^0 \]

\[ V = \{ (z, w) \in \Sigma \mid z \neq 0 \} \]

\[ \phi : U \rightarrow V, \quad \phi(x,y) = (\frac{1}{y}, \frac{1}{x^{g+1}}) \]

\[ \mathbb{Z} = \Sigma \amalg \Sigma^0 \]

Check: \( \mathbb{Z} \) is a compact R.S.

\[ \text{cln } U : \]

\[ \downarrow \pi \]

\[ \times \]

\[ \pi : \mathbb{Z} \rightarrow \mathbb{P}^1 \quad (\text{Riemann sphere}) \]

\[ \mathbb{C}^\infty \]
Claim: \( g(2) = g. \)

\[ \text{Pf: } \quad 2g(2) - 2 = 2(-g) + \sum_{p \in \mathbb{Z}} |\text{mult}_p - 1| \]
\[ = -4 + \deg g + \sum_{\text{clg h, has many zeros}} \]
\[ = \deg g - 2. \]
\[ \Rightarrow g(2) = g. \]

Recall: \( w \) mean 1-form on compact R.S. \( \mathcal{X} \). Then \( \# \text{zeros} - \# \text{poles} = \text{const.} \)

Claim: \( \mathfrak{g} \) compact R.S., \( \mathcal{X} \) has a mean. ftn \( f: \mathcal{X} \to \mathbb{C} \) (by assumption)
and \( f \) is unramified at \( \infty \) (makes \( \infty \) only have to deal w/ 1 case).
Then if \( w \) is a mean 1-form on \( \mathcal{X} \), \( \deg w = \deg g - 2. \)

\[ \text{Pf: } \quad f \text{ gives rise to } F: \mathcal{X} \to \mathbb{P}^1: \mathbb{C}^\infty, F \text{ holomorphic.} \]
\[ \text{Local coordinate in } \mathbb{C}^\infty \text{ in part that contains } 0. \]
\[ w = dz. \text{ } \]w has no zeroes. \( \text{w does have poles, they are at } \infty \). \( \mathcal{X} \to \frac{1}{w} \).
\( \text{In the other coordinate patch, } w = d(z) = \frac{1}{w} \), \( \frac{1}{w} \) has a pole of order 2 at \( \infty \).
\[ \text{Pull \ } w \text{ back to } \mathcal{X}: \]
\[ \eta = F^* w, \text{ mean 1-form on } \mathcal{X}. \]

\[ \text{Count zeros and poles:} \]
\[ \text{locally: } \quad \begin{array}{c}
\mathfrak{g} \in \mathcal{X}, \text{ root ramification pt., } \mathfrak{g} \nearrow \infty.
\text{Local coord. } x \text{ centered at } p. \quad F: x \to x = z.
\end{array} \]
\[ F^* dx = dx. \text{ No zeros, no poles.} \]
\[ \begin{array}{c}
\mathfrak{g} \in \mathcal{X}, \text{ p ram pt. } \mathfrak{g} \nearrow \infty.
\text{Local coord. centered at } p. \quad F: x \to x = -z
\text{for min } m. \quad F^* dx = d(x^m) = m x^{m-1} dx.
\end{array} \]
so we have \( m-1 \) zeros for each ramification pt.

\[ m = \text{mult}_p F. \]

\[ \text{pt} \in \mathbb{P}^1, \quad \text{pt} \to \infty. \]

\[ uw \text{ at } \text{pt}, \quad x \mapsto x = w \quad (\text{by assumption}). \]

\[ F'' (-\frac{1}{2} x u) u = -\frac{1}{x^2} dx \]

2 poles for each point \( p \to w \)

\[ \# \text{poles} = 2 \deg F. \]

\[ \# \text{zeros} - \# \text{poles} = \sum_{p \in \mathbb{P}^1} [\text{mult}_p F - 1] = 2 \deg F \]

\[ = 2g - 2 \]

**Useful Lemma:** Let \( \mathbb{X} \) be a smooth affine curve in the plane defined by \( f(x, y) = 0 \).

Define \( \pi : \mathbb{X} \to \mathbb{C} \) by \( \pi(x, y) = x \). Then \( \pi \) is ramified at \( p \in \mathbb{X} \) iff

\[ (\frac{\partial \pi}{\partial y})(p) = 0 \]

Similarly, \( \mathbb{Y} \) is a projective plane curve defined by \( F(x, y, z) = 0 \).

Define \( \pi : \mathbb{Y} \to \mathbb{P}^1 \) by \( \pi(x, y, z) = (x : y) \). Then \( \pi \) is ramified at \( p \in \mathbb{Y} \) iff

\[ (\frac{\partial \pi}{\partial z})(p) = 0 \]

**Example:** Consider the Fermat curve \( x^d + y^d + z^d = 0 \in \mathbb{P}^2 \). Then

\[ \frac{\partial \pi}{\partial z} = d \quad \text{at zero if } z = 0, \text{ i.e., } x^d + y^d = 0. \]

Then the pt \( [x : y : 0] \) is ramified if \( x^d y^d = 0 \), i.e., the points \( [1 : y : 0] \) where \( y^{d+1} = 0 \) are the ramification pts. Then

\[ 2g(\mathbb{X}) - 2 = (2g(\mathbb{Y}) - 2) \deg \pi + \sum \text{mult}_p F - 2 \]

\[ = (2d + 1) - 2d + (d-1) = 2d + d(d-1). \]

ie,

\[ g(\mathbb{X}) = \frac{(d-1)(d-2)}{2}. \]
Compute the genus of the curve $y^2 = x^3 + 1$.

Let $F = y^2 z - x^3 - z^3$, $x = (F = 0)$. Then define a map $\pi : \mathbb{A} \to \mathbb{P}^1$ by $[x:y:z] \mapsto [x:y:1]$. Note this is well defined because $x$ and $y$ can't both be zero since $y^2 = x^3 + 1$. The ramification points occur precisely when $\frac{\partial F}{\partial z} = 0$, i.e., when $y^2 - 3xz^2 = 0$. If $z = 0$, then $y = 0 \Rightarrow x = 0$ if we are on $x$. So $z \neq 0$, so we can set $z = 1$.

Thus the ramification points are when $y^2 = 3$ i.e., $y = \pm \sqrt{3}$.

Hence $3 - 1 = x^3$, i.e., $x = \omega^i \sqrt{3}$, so there are 6 ramification points, each with multiplicity 2 since $\frac{\partial F}{\partial z}$ has degree 2.

Then for any point $[x:y]$, there are 3 points sitting over it, corresponding to the solutions of $y^2 z - x^3 - z^3 = 0$.

Thus deg $\pi = 3$. As R.H. gives,

$$\theta g(\mathbb{A}) - 2 = 3 \cdot (-2) + \sum_{p \mid \mathbb{A}} (2 - 1)$$

$$= -6 + 6 = 0 \Rightarrow$$

$g(\mathbb{A}) = 2$. 

Let $F: X \to Y$ be a monomorphism holomorphic between compact R.S.

1. Show that if $Y \cong \mathbb{P}^1$, and $F$ has deg $> 1$, then $F$ must be ramified.

**Pf:** \( g(Y) = 0 \), so R.H needs

\[ \Sigma g(X) - 2 = (\deg F) (\deg Y - 1) + \Sigma (\text{mult}_F - 1). \]

Suppose $F$ is not ramified, i.e. \( \Sigma (\text{mult}_F - 1) = 0 \).

Then \( \Sigma g(X) - 1 = -\deg F , \) i.e.

\[ 1 - g(X) = \deg F . \]

because \( \deg F \geq 0 \) and \( g(X) \geq 0 \).

2. Show that if both $X$ and $Y$ are genus 1 then $F$ is unramified.

**Pf:** R.H needs

\[ 0 = 0 + \Sigma (\text{mult}_F - 1) \]

Since \( \text{mult}_F \geq 1 \) always, we must have \( \text{mult}_F = 1 \) everywhere.

3. Show that \( g(X) \geq g(Y) \) always.

**Pf:**

\[ \Sigma g(X) - 2 = \deg F (\deg Y - 2) + \Sigma (\text{mult}_F - 1) \]

\[ \geq \deg F (\deg Y - 1) \]

\[ \geq \deg Y - 2 \quad \text{if} \quad \deg Y - 2 \geq 0 \]

\[ \Rightarrow g(X) \geq g(Y) \quad \text{if} \quad g(Y) \geq 1. \]

But if \( g(Y) = 0 \), \( g(X) \geq 0 \) automatically.
Show that if \( g(y) = g(z) \geq 0 \), then \( F \) is an isomorphism.

**Proof:**

R.H. needs:

\[
2g - 2 = \deg F (2g - 2) + \sum (\text{mult} F - 1).
\]

\[
(2g - 2)(1 - \deg F) = \sum (\text{mult} F - 1).
\]

\[g \geq 0 \Rightarrow 2g - 2 \geq 0 \Rightarrow 1 - \deg F \geq 0\]

\[\Rightarrow \deg F = 1\]

\[\Rightarrow F \text{ is an isomorphism.}\]
Example: Compute the genus $g$ of the Fermat curve $x^d+y^d+z^d=0$.

Let $x^d+y^d+z^d=F$ and $F=0 \implies \mathcal{E}$. Let $\mathcal{Y}=\mathbb{P}^1$ and define

$\pi: \mathcal{E} \to \mathcal{Y}$ by $[x:y:z] \mapsto [x:y]$. 

Now $\deg \pi = d$ because $\deg F = d$.

The ramification pts are where $\frac{\partial F}{\partial z} = 0$, i.e. when $z = 0$. So they are the points where $x^d+y^d=0$, i.e., $[1:y:0]$ where $y^d = -1$.

Then these are $d$ ramification points.

To compute the multiplicity write

$[x:y:z] \mapsto [x:y]$. 

So there are $d$ points going to each branch point, as the result is $d$.

Thus, N.H. reads

$2g(\mathcal{E}) - 2 = d(-2) + d(d-1)$

$\therefore \quad g(\mathcal{E}) = \binom{d-1}{2}.$
\text{Last lines:}

\( f(x) \) is represented locally as \( z^k \), we call \( (c-1) \) the ramification index \( e_{(x)} \).

\( R = \{ x \mid r(x) \geq 1 \} \) ramif. locus.

\( \mathbb{R} \), \( \mathbb{Y} \) compact. Then \( R, f(R) = \mathbb{B} \) are finite.

Then \( \mathbb{R} \setminus f^{-1}(\mathbb{B}) \to \mathbb{Y} \setminus \mathbb{B} \) is a covering space, and

\[ \text{deg } f = \sum_{x \in \mathbb{R}} e_{(x)}. \]

\text{Corollary:} If \( \mathbb{R} \) is a compact R.S which carries a non-constant meromorphic function, then for any meromorphic 1-form \( \omega \) on \( \mathbb{R} \),

\[ \sum_{x \in \mathbb{R}} \text{ord}_x (\omega) = 2g - 2, \]

where \( g = g(\mathbb{R}). \)

\text{Holomorphic and Meromorphic 1-forms on Plane Curves:}

Suppose \( \mathbb{R} = \{ f(x, y) = 0 \} \in \mathbb{C}^3 \) is a nonsingular plane curve.

Keep in mind \( x, y \) are holomorphic coordinates, not real and imaginary parts!

\text{Prop:} Consider meromorphic 1-forms \( \frac{dx}{dy}, \frac{dy}{dx} \) on \( \mathbb{C}^2 \). On \( \mathbb{R} \),

\[ \left. \frac{dx}{dy} \right|_{\mathbb{R}} = -\frac{\frac{dy}{dx}}{2\frac{\partial f}{\partial x}} \mid_{\mathbb{R}} \] and they are holomorphic on \( \mathbb{R} \).

\text{Proof:} That they are holomorphic follows from \((*)\), since one of \( \frac{dx}{dx}, \frac{dy}{dy} \) is non-vanishing at each point on the curve.
\[
\frac{df}{dx} dx + \frac{df}{dy} dy,
\]
\[
\frac{df}{f(x,y)} = 0 \quad \text{clearly. But} \quad \{ f = 0 \} = \mathbb{R}, \quad \text{so we get the result.}
\]

**Goal:** More generally, for any poly \( p(x,y), \)

\[
p(x,y) \frac{dx}{dx / dy} = u_p
\]

is a **hole 1-form on \( \mathbb{R} \).**

**Classical Example:**

\[
\frac{dx}{\sqrt{4x^2 + 9y^2 + 9z^2}} = \frac{dx}{y} \quad \quad y^2 - 4x^3 - 9x - 9z = 0
\]

**Now consider the smooth curve** \( \mathbb{R} \subseteq \mathbb{R}^3 \)

\[\mathbb{R}_u \subseteq \mathbb{R}^3\]

**When do \( u_p \) extend to hole forms on \( \mathbb{R} \)?**

**Set-up:**

Consider \( \mathbb{R} \subseteq \mathbb{R}^3 \) non-sing curve \( x = dy \) e. defined by homy, deg p

\[H(x, y, z) = 0\]

**Local coordinates:**

\[
x = \frac{X}{2} \quad s = \frac{Y}{X}
\]

\[
y = \frac{Y}{X} \quad t = \frac{Z}{X}
\]
Local Equation for curve:

\[ f(x, y) = H(x, y, z) \quad g(s, t) = H(1, s, t) \]

Transitions:

\[ s = \frac{x}{2}, \quad t = \frac{1}{x} \]
\[ x = \frac{1}{t}, \quad y = \frac{2}{x} \]

\[ f(x, y) = H\left(\frac{1}{t}, \frac{2}{x}, z\right) \]
\[ = \frac{1}{x} H\left(1, s, t\right) \]
\[ = \frac{1}{x} g(s, t) \]

\[ g(s, t) = x e g\left(\frac{x}{2}, \frac{t}{x}\right) \]

\[ \frac{\partial g}{\partial s} = t e \frac{\partial f}{\partial y}\left(\frac{1}{t}, \frac{2}{x}\right) \]
\[ = t e \frac{\partial f}{\partial y}\left(\frac{1}{t}, \frac{3}{x}\right) \]

Start with:

\[ \omega_p = p(x, y) \frac{\partial f}{\partial y} \]

and transform into \( s, t \) coordinate.

\[ \omega_p = p\left(\frac{1}{t}, \frac{2}{x}\right) \left(-\frac{\partial f}{\partial x}\right) \]
\[ = \frac{1}{t e^{-1}} \cdot \frac{\partial g}{\partial s}\left(s, t\right) \]
\[ = \left(\frac{1}{x} p\left(\frac{1}{t}, \frac{2}{x}\right) \cdot t e^{-3}\right) \frac{dt}{ds} \]

When is this whole?

Need \( t e^{-3} p\left(\frac{1}{t}, \frac{2}{x}\right) \) to be holomorphic.
It is sufficient that $\deg p \leq e - 3$.

Upshot (modulo checking another coordinate patch):

$W = p(x, y) \frac{dx}{dy}$ extends to a holomorphic form on $\nu$.

$p(x, y) \leq 1$ (deg $e$) provided that $\deg(p) \leq e - 3$.

**Thm:** Let $\Sigma \subset \mathbb{P}^2$ be a smooth curve of degree $e$. Then

- Strong form of?
- $\deg = e - 3$

\[\text{LS}\]

- Poly of degree $e$?
- $e - 3$ in $xy$?

\[\xrightarrow{\text{ hole 1-forms?}} \xrightarrow{\text{ on } \Sigma}\]

$p(x, y) \rightarrow \left( p(x, y) \frac{dx}{dy} \right) \{f = 0\}$ every for $\Sigma \subset \mathbb{C}^2$.

This is actually a special case of the "Adjunction Formula" (see Hart).

**What about curves w/ singularities?**

$\Sigma_0 \subset \mathbb{C}^2$ a possibly singular curve, $\Sigma_0 = \{ f(x, y) = 0 \}$.

**Thm:** Exists smooth 1-dimensional variety $\Sigma$ plus a finite birational map $\nu: \Sigma \to \Sigma_0$.

\[\xrightarrow{\nu} \]

\[\Sigma_0 \to \Sigma\]
Let \( w_p = p(x,y) \frac{dx}{dy} \)

**Ask:** When is \( v^y(w_p) \) held on \( \mathcal{A} \)?

In general, \( p(x,y) \) has to vanish appropriately at singular points of \( \mathcal{A}_0 \) in order to cancel poles from \( \frac{dy}{dx} \).

The vanishing condition on \( p(x,y) \) is a delicate invariant of singularity of \( \mathcal{A}_0 \).

We’ll need the case when \( \mathcal{A}_0 \) has ordinary double points.
Recall: \( \mathbb{A} \subset \mathbb{P}^3 \) is nonsmooth plane curve of degree \( d \),
\[ \{ \text{homo. poly. deg-3} \} \longrightarrow \{ \text{holo 1-forms} \} \] on \( \mathbb{P} \).

In affine coordinates,
\[ p(x,y) \rightarrow p(x,y) \frac{dx}{\partial x} \]
\[ \text{deg} = 3 \]

What is the condition on \( p(x,y) \) in order that \( \psi^\ast (\frac{dx}{\partial y}) \) be holo on \( \mathbb{P} \)?

**Special Case:**

Any \( \mathbb{A} \) has only ordinary double point at \( p \):

\[ \begin{align*}
\infty & \quad \quad \text{X.} \\
\text{locally} : \quad \mathbb{A} &= \{ xy = 0 \}.
\end{align*} \]

**Example:** \( y^3 - x^7(x+1) = 0 \)

Locally, \( \mathbb{A} \) has two branches:

\[ f = xy, \]

\[ \begin{array}{c}
V_1 \\
(5,0) \\
V_2 \\
(10,5)
\end{array} \]

\[ \psi_p = p(x,y) \frac{dx}{\partial y} = - p(x,y) \frac{dy}{\partial x} \]

\[ V_1^\ast (\psi_p) = p(t,0) \frac{dt}{\partial t} \quad \text{Need} \quad p(0,0) = 0. \]

\[ V_2^\ast (\psi_p) = - p(0,s) \frac{ds}{\partial s} \]

If \( \mathbb{A} \) has ordinary double point, then \( V^\ast (\psi_p) \) is holo provided that \( p \) vanishes at the double point.
Thm: Let $\overline{X} \subseteq \mathbb{P}^3$ be a curve of degree $d$ with only ordinary double points, $\nu: X \to \overline{X} \subseteq \mathbb{P}^3$ desingularizing. Then

\[
\begin{align*}
\{ \text{homog polynomials} \} & \to \{ \text{holo 1-forms} \} \\
\{ \text{deg} e-3 \text{ vanish at double pts of } \overline{X} \} & \to \{ \text{1-forms on } X \}
\end{align*}
\]

\[
P(x,y) \mapsto \nu^*(\rho \frac{dx}{dt/\partial y}).
\]

**Realizing R.S.'s as Plane Curves with Nodes:**

**Convention:** From now on, algebraic curve will mean non-singular complex projective variety of dim 1.

**Fact:** Any compact R.S. is an alg. curve, and a holomorphic morphism between compact R.S.'s is an algebraic map.

**Thm:** Let $X \subseteq \mathbb{P}^r$ be a smooth alg. curve.

1. $X$ can be embedded in $\mathbb{P}^3$.
2. $f$ birational map $\nu: \overline{X} \to X$ of $\mathbb{P}^3$ from $\overline{X}$ onto a plane curve $X$ having only ordinary double points.

**Older Proofs:** In each case, map will arise as linear projection w center $L \subseteq \mathbb{P}^r$.

\[
\dim L = \begin{cases} 
\frac{r-4}{2} & \text{case 1} \\
\frac{r-3}{2} & \text{case 2} 
\end{cases}
\]

\[
\begin{bmatrix}
\{ \text{L} \} = \{ T_0 = \cdots = T_3 = 0 \}.
\end{bmatrix}
\]

\[
\nu_L: \mathbb{P}^r \setminus L \to \mathbb{P}^3
\]

\[
\nu_L([x_0, \ldots, x_r]) = [x_0, \ldots, x_3]
\]
For $0$: Starting with curve $X \subseteq \mathbb{P}^r$, we'll argue that for a suff. general choice of $L$, $\pi_L|_X : X \to \mathbb{P}^3$

Take $X \cap L = \emptyset$, so $\pi_L|_X$ is defined. Then $\pi_L$ is an embedding iff

* $\pi_L$ is 1-1

* $d\pi_L|_X$ is 1-1 at every point of $X$.

**Note:**

$\pi_L(x) = \pi_L(y)$

iff line $\overline{xy}$ joining $x$ and $y$ meets the center of projection $L$.

Need to choose $L$ so as to avoid all lines joining all pairs of points of $X$.

Consider

\[ \text{Sec}(X) = \{ \text{Zariski closure of all lines } \overline{xy} \mid x, y \in X, \text{ distinct pts} \} \]

**Claim:** $\text{Sec}(X)$ is an irreducible variety of dim $\leq 3$.

**Proof:** Locally,

\[ \text{Sec}(X) \subseteq X \times X \times \mathbb{P}^1 \]

each piece of $X$

**Upshot:** Need to choose $L \cap \text{Sec}(X) = \emptyset$ for $\pi_L|_X$ to be 1-1.

$\mathbb{P}^r$ dim $L = r-4$ so this happens just by counting these dimensions.

**Claim:** Automatically $\pi_L|_X$ has non-ram deriv.

Since Sec($X$) is Zariski closed, it contains the tangent lines (limit of secant lines), so we miss tangent line as well.
Fin $2$: Start w/ $X \in \mathbb{P}^3$ (by 0), and project from a point.

- Need to show that a general point in $\mathbb{P}^3$ lies in only finitely many secant lines. (ok from dim).
- Need to show not every secant line is a tri-secant line, i.e., has curve three times.
- Control tangent lines at secant points.
Bézout's Thm:

Thm: Let $E_1, E_2 \subseteq \mathbb{P}^2$ be curves of degrees $d, e$ defined by homogeneous polynomials $F_1, G_2$. Assume $F_1, G_2$ have no common factor. Then $\#(E_1 \cap E_2) < \infty$ and $E_1 \cap E_2$ consists of $d \cdot e$ points counting multiplicities.

Multiplicities:

\[ \begin{array}{c}
\chi \\
\gamma
\end{array} \]

Fix $p \in E_1 \cap E_2$. In the local ring $\mathcal{O}_p(\mathbb{P}^2)$, consider local equations $f, g \in \mathcal{O}_p(\mathbb{P}^2)$. Then $i_p(E_1, E_2) = \dim \frac{\mathcal{O}_p(\mathbb{P}^2)}{(f, g)}$.

Thm (restated): In situation of thm, $\sum_p i_p(E_1, E_2) = d \cdot e$.

Example: $f = y$, $g = y - x^n$

\[ \frac{\mathbb{C}[x, y]}{(y, y - x^n)} = \frac{\mathbb{C}[x, y]}{(y, x^n)} = \frac{\mathbb{C}[x]}{(x^n)} \quad \text{by dim n.} \]

\[ i_p(f, g) = n. \]

Remark: The essential content of the thm is that $\sum_p i_p(E_1, E_2)$ only depends on $d, e$. 
There is a very short proof of this using cohomology.

We'll prove a special case when \( \mathcal{X} \) is smooth.

**Lemma:** In situation of this, let \( \hat{O}_p(\mathcal{X}) = \hat{O}_p(\mathcal{X}, y) \) be the completion of \( \mathcal{O}_p(\mathcal{X}) \). Let \( \mathcal{O}_p(\mathcal{X})^\omega = \mathcal{O}_p(\mathcal{X}, y) \) be the ring of convergent power series. Let \( \hat{f}, \hat{g} \in \hat{O}_p(\mathcal{X})^\omega \), \( f, g \in \mathcal{O}_p(\mathcal{X})^\omega \) be the corresponding local equations. Then

\[
\hat{f}/(f, g) \cong \mathcal{O}_p/(f, g) \cong \mathcal{O}_p/(f, g)
\]

In particular, have same dimensions as \( \mathcal{O}_p \)-v.s.'s.

**Claim:** \( \mathcal{I}/m^n \cong \hat{\mathcal{O}}^\omega/m^n \cong \mathcal{O}_p/(x, y)^n \cong \mathcal{O}_p/(x, y)^n \)

By Nullstellensatz, since \( f, g \) contain \( p \) as an isolated intersection point, \( \mathcal{I}/(f, g) = m \Rightarrow m^n \subseteq (f, g) \), similarly in \( \mathcal{O}_p, \mathcal{O}_p/(x, y)^n \).

**Lemma:** In situation of this, let \( \mathcal{X} \) be non-sing. \( g = \) local eqn for \( \mathcal{X} \) near \( p \). Then

\[
i_p(\mathcal{X}, g) = \text{ord}_p(g|_{\mathcal{X}})
\]

At points \( g\text{-}\text{val} = \text{ht}_m \text{ on } \mathcal{X}\)

(This makes sense since \( \mathcal{X} \) is smooth).

**Pf:** Choose local analytic parameterization for \( \mathcal{X} \) near \( p \):

\[
\mathcal{X} = \bigsqcup_{\lambda} \bigsqcup_{\beta} x_{\lambda \beta}(z_1 \ldots z_\alpha)
\]
Then \( \frac{\partial^n f}{\partial z^n} \cong c z^2 \int f = ax + by + \text{higher order terms}, \quad b \neq 0, \quad b = 1. \)

\[
\frac{\partial^n f}{f^n} = \frac{C \{ x, y \}}{(y + ax + bxz)} \cong C \{ x \}
\]

\[
\frac{\partial^n f}{f^{n+1} g} = \frac{C \{ z \}}{g(z, \theta z)} \quad \dim = \text{ord}_p (g(z)). \quad \square
\]

**Proof:** (when \( \mathcal{E} \) is smooth) let \( y, y' \in \mathbb{P}^2 \) be 2 curves \( \mathcal{E} \) of degree \( \delta \),

with equations \( G, G' \) having no common factor \( F \).

By Remark above, it is enough to show

\[
\sum i_p(\mathcal{E}, y) = \sum i_p(\mathcal{E}, y').
\]

To show it actually is true, can just look at branches intersecting. We only need to show it only depends on \( \delta \) and \( \mathcal{E} \).

For this, consider \( \mathcal{G} \mid \mathcal{E} \) a meromorphic map on \( \mathcal{E} \).

By 2nd lemma,

\[
\sum i_p(\mathcal{E}, y) - \sum i_p(\mathcal{E}, y') = \sum \text{ord}_p (g) - \sum \text{ord}_p (g') \quad (9.3', \text{local on } \mathcal{E})
\]

\[
= \sum \text{ord}_p (9g')
\]

\[
= 0 \quad (\text{because maps from have same } \delta)
\]

\[
\mathcal{E}, \mathcal{G}, \mathcal{G}'
\]
**Genus Formula:**

**Thm:** Let \( C \) be a smooth plane curve of degree \( d \). Then

\[
g(C) = \binom{d-1}{3} = \frac{(d-1)(d-2)}{2}
\]

\[\iff d-g-2 = d(d-3)\]

<table>
<thead>
<tr>
<th>( d )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

This shows you miss a lot of generic using only smooth plane curves.

(No smooth plane curve genus \( \geq 5 \) for example)

**Fact 4:** \((d=3)\)

**Recall:**

\[
(d-g-2) = \sum_{p \in C} \text{ord}_p (w), \quad w \text{ a mer 1-form}
\]

**Recall:** \(Q \) = homog poly of deg \( d-3\)

\[
\text{Hilb 1-form } \frac{Q(x,y)}{w_0} = \frac{q(x,y)}{\partial F/\partial y} \quad \left( F=0 \right) = \mathbb{R}
\]

**Claim:** \( \text{ord}_p (w_Q) = i_p(\mathbb{R}, \sigma = 0) \).

**Sketch:** \( \frac{dx}{\partial y} \) is vanishing and \( \text{hull in } \mathbb{R} \).

\( \text{ord} (w_Q) = \text{ord}_p (Q) = i_p(\mathbb{R}, \sigma = 0) \).
Now by Bezout:

\[ \sum \phi(\mathfrak{p}, \mathfrak{q}) = d(d-3) \]

\[ \phi^2 -a \]

\[ \phi \geq 0 \]
Divisors:

Let \( \mathcal{E} \) = compact R.S. (or smooth projective).

**Def:** A divisor on \( \mathcal{E} \) is a finite formal \( \mathbb{Z} \)-lin combination of \( p \). All such is an additive abelian group \( \text{Div}(\mathcal{E}) \).

**Notation:** \( D = \sum n_p \cdot p \), \( n_p \in \mathbb{Z} \), all but finitely many are 0.

or \( D = \sum n_p \cdot p \).

**Example:** \( 0 \neq f \in \mathcal{E}(\mathcal{E}) \).

\[
\text{div}(f) = \sum \text{ord}_p(f) \cdot p.
\]

- Has a zero at \( p \) iff \( \text{ord}_p(f) \) is negative.
- Has a pole at \( p \) iff \( \text{ord}_p(f) \) is non-zero.

Similarly, if \( \omega \) is a non-zero meromorphic 1-form, then

\[
\text{div}(\omega) = \sum \text{ord}_p(\omega) \cdot p.
\]

**Def:** The degree of \( D \) is given by \( \text{deg}(D) = \sum n_p \).

**Example:** if \( 0 \neq f \in \mathcal{E}(\mathcal{E}) \), \( \text{deg}(\text{div}(f)) = 0 \), \( \# \text{zeros} = \# \text{poles} \).

Similarly, \( \text{deg}(\text{div}(\omega)) = 2g - \delta \), where \( g = g(\mathcal{E}) \).

**Def:** A divisor \( D \) is effective if \( \text{ord}_p(D) \geq 0 \) for all \( p \in \mathcal{E} \).

**Notation:** \( D \geq 0 \), \( (D \geq 0) \).

**Def:** Given divisor \( D \), define

\[
\mathcal{L}(D) = \{ f \in \mathcal{E}(\mathcal{E}) \mid \text{div}(f) + D \geq 0 \}, \quad \text{or} \{ f \}.
\]
$L(D)$ is a $C$-v.s.

**Explanations:**

- Any $D = P_1 + \cdots + P_r$ distinct points.
  \[ f \in L(D) \iff \text{div}(f) + \sum P_i \geq 0 \]
  \[ \iff \text{div}(f) \geq -\sum P_i. \]
  \[ \iff f \text{ has at worst a simple pole at } P_i \text{ and no other poles.} \]

- Similarly, any $D = \sum n_i P_i$, $n_i > 0$. (effective divisor).
  \[ f \in L(D) \iff f \text{ has at worst a pole of order } n_i \text{ at } P_i \text{ and no other poles.} \]

- Any $D = \sum n_i P_i - \sum m_j Q_j$, $n_i, m_j > 0$
  \[ f \in L(D) \iff \text{div}(f) + \sum n_i P_i - \sum m_j Q_j \geq 0 \]
  \[ \iff \text{div}(f) \geq -\sum m_j Q_j - \sum n_i P_i. \]
  \[ \iff f \text{ has at worst a pole of order } n_i \text{ at } P_i \text{ and no other poles and zeros of order greater than } \]
  \[ \text{equal to } m_j \text{ and no other zeros.} \]

"$L(D)$" allows poles and requires zeros.

**Example:** $\mathbb{X} = \mathbb{C}/\Lambda$, $\delta \in \mathbb{X}$ the origin.

$V_\delta = L(\delta, \delta).

**Thm:** $L(D)$ is a finite dimensional $C$-v.s. and in fact,
\[ \dim L(D) \leq \deg D + 1. \text{ If } \deg D < 0, \text{ then } L(D) = 0. \]
Example: $X = P^1$, $D = K \cdot 0 \in C^2$

$$L(D) = \{ \text{polys of degree } n \}$$

Thus, $\dim L(D) = K + 1 = \deg D + 1$.

Proof (Thm): If $\deg D < 0$ and $f \in L(D)$, then $\deg f + D > 0$.

But $\deg(f) = 0$, $\deg D < 0$, so $\deg (LHS) < 0$. $\#$.

Note that if $D' \geq D$, then $L(D) \subseteq L(D')$.

To prove the theorem, it is enough to show $\forall D$, for any $P \in X$,

$L(D) \subseteq L(D + P)$ has codim = 1. (\#)\#.

Suppose $0 \neq f, g \in L(D + P)$, any $\eta_p = \text{ord}_p(D)$, and $f, g \in L(D)$.

Assume for clarity $\eta_p > 0$.

Since $f, g \notin L(D)$, $\text{ord}_p(f) = \text{ord}_p(g) = -\eta_p - 1$.

Take a local coordinate centered at $P$, then

$$f = \frac{a_0}{x^{\eta_p}} + \text{higher order terms}$$

$$g = \frac{b_0}{x^{\eta_p}} + \text{higher order terms}$$

As $\text{ord}_p(bf - ag) \geq -\eta_p$. Thus $bf - ag \in L(D)$,

i.e., $bf \equiv ag \pmod{L(D)}$.

Plan: Take basis $f_0, \ldots, f_r \in L(D)$. Define $X \rightarrow P^r$

$$(z_1 \rightarrow [f_0(z_1), \ldots, f_r(z_1)])$$

The Riemann-Roch problem is to actually compute the dimension of $L(D)$. 
Linear equivalence of Divisors:

Def: Two divisors \( D, D' \) on \( X \) are linearly equivalent \( D \equiv D' \) if \( D - D' = \text{div}(f^i), 0 \neq f \in \mathcal{O}(X) \).

Def: Principal divisors = \( \text{Princ}(X) = \{ \text{div}(f(x)) : 0 \neq f \in \mathcal{O}(X)^* \} \cup \{0\} \).

\( \text{Princ}(X) \subseteq \text{Div}(X) \), as a subgroup. \( \text{div}(f(x) + f(y)) = \text{div}(f(x)) \).

Def: Divisor class group \( \text{Cl}(X) \) (or \( \text{Pic}(X) \)) is \( \text{Cl}(X) = \frac{\text{Div}(X)}{\text{Princ}(X)} \).

Prop: \( \text{Cl}(D) = \text{Cl}(D') \), then \( \mathcal{L}(D) \cong \mathcal{L}(D') \).

Prop: Next class.

Exercise: \( \text{Cl}^0(X) = \frac{\text{Div}^0(X)}{\text{Princ}(X)} \). \( \text{Div}^0(X) = \text{divisors of degree 0} \).

Describe \( \text{Cl}^0(X) \) when \( X = \mathbb{C}/\Lambda \).

(Abel's Thm).
Example: Consider a holomorphic map \( \mathbb{P} \to \mathbb{P}^r \) "ram. deg." in the sense \( \Phi(x) \notin \text{any hyperplane}. \)

Given a hyperplane \( H \subseteq \mathbb{P}^r \), define intersection divisor

\[
\mathbb{X} \cdot \Phi H = \sum_{P \in \mathbb{X}} i_P P
\]

\( i_P \): choose local affine eq \( H \)

\( d_P P = \Phi^{-1}(H) \), define \( i_P = \text{ord}_P(\Phi^{-1}(H)) \)

Exercise: \( \text{deg}(\mathbb{X} \cdot \Phi H) \) is indep of hyperplane \( H \).

Remark: Can do this replacing \( H \) by any hypersurface \( G \) of deg \( e \)

(If \( G \) doesn't vanish identically in \( \Phi(x) \)).

Let \( \mathbb{X} \) be a projective curve, fix \( D \) on \( \mathbb{X} \). We defined \( D(\mathbb{D}) \) last time.

We include \( 0 \) in \( D(\mathbb{D}) \) even though it doesn't have a divisor.

Example: if \( D \neq 0 \), then \( 1 \in D(\mathbb{D}) \).

Eventually: We'll analyze \( \mathcal{O}(\mathbb{X}) = \text{Div}^0(\mathbb{X}) \) where \( \text{Div}^0(\mathbb{X}) = \text{divisors of degree } 0 \).
Example: \( \omega = \text{mero} \quad \text{b} = \text{one} \quad \text{form} \)

\[ K = \text{div}(\omega). \]

\( K \) is called a canonical divisor.

1. \( \text{div}(\omega), \text{div}(\omega') \) are two such \( \omega \) divisors \( K, K' \), then \( K = K' \).
2. \( \mathcal{L}(K) \cong \{ \text{line} \_ \text{forms} \} \)

**Pf:**

1. \( \text{div}(\omega) - \text{div}(\omega') = \text{div}(\frac{\omega'}{\omega}) \)
2. \( \mathcal{L}(K) \mapsto \{ \text{line} \_ \text{forms} \} \)

\[ f \mapsto f \omega \]

\( f \in \mathcal{L}(K) \) iff \( \text{div}(f) + K = 0 \)

\[ \text{div}(f) + \text{div}(K) = 0 \]

\( \text{div}(f) + 0 \rightarrow f \omega \) hold

**Prop:** 
\( \text{div}(D) = 0 \), then \( \mathcal{L}(D) \cong \mathcal{L}(D') \).

**Pf:**

\( \text{div}(D) = \text{div}(\phi) \).

\[ \mathcal{L}(D) \mapsto \mathcal{L}(D') \]

\[ f \mapsto \phi \ f \]

Check this works, it is straightforward from the def.

**Def (Exercises):** Let \( D \) be any divisor on \( \mathcal{X} \). The complete linear series associated to \( D \) is

\[ |D| = \{ \text{effective divisor } \ D', \ s.t. \ D = D' + \mathcal{T} \} \]

\[ \mathcal{E} = \{ \text{div}(f) + D \mid f \in \mathcal{L}(D) \} \}

\[ \cong \mathbb{P}(\mathcal{L}(D)) \]
Exercise: Assume $D$ smooth, choose a basis $1, f_1, \ldots, f_r$ of $\mathcal{L}(D)$.

Let $\phi = \phi_{D_1} : \mathbb{P}^r \to \mathbb{P}^r$. Assume $D' \in D_1$ do not contain any common point.

Then $D_1 = \{H \in \mathbb{P}^r \mid H \cap \phi(H) \neq \emptyset\}$.

More sophisticated: let $\Delta$ be a divisor on $\mathbb{P}^r$ with linear bundle $\mathcal{L}_\Delta(D)$ plus maps. Section $\gamma$ by $\mathcal{L}_\Delta(D)$.

$$D = D' \Leftrightarrow \mathcal{L}_\Delta(D) \cong \mathcal{L}_\Delta(D').$$

$$\mathcal{L}(D) \cong \mathcal{L}_\Delta(D).$$

Riemann's inequality: let $D$ be any divisor on $\mathbb{P}^r$ (smooth proj. curve), then $\dim \mathcal{L}(D) \geq \text{deg } D + 1 - g(\mathbb{P}^r)$.

Example: $\mathbb{P}^r = \mathbb{C}/\Delta$, $V_K = \mathcal{L}(K \cdot \overline{\Delta})$.

Riemann's Thm says:

$$\dim V_K \geq K + 1 - 1 = K.$$
Linear Systems of Plane Curves:

Consider \( \mathbf{V}_d = \{ \text{all homog. poly. deg. } d \in \mathbb{R}^n \} \)
\[ = S^d (\mathbb{V}, 1) \]
Complex vector space, \( \text{dim} \mathbf{V}_d = \binom{d+2}{2} \).

Can view \( \mathbb{P}(\mathbf{V}_d) \) as space to all plane curves of degree \( d \).

**Example:** Conics in \( \mathbb{P}^2 \) is \( \mathbb{P}^5 \).

Write "general" \( F \in \mathbf{V} \) as \( \sum_{i+j+k=d} t_{ijk} x^i y^j z^k \). The coefficients \( t_{ijk} \) are the natural coordinates; \( t_{ijk} \) are natural coordinates in \( \mathbf{V}_d \).

**Lemma:** Fix \( p \in \mathbb{P}^2 \). Then the set \( \mathbf{V}_d (p) = \{ F \in \mathbf{V}_d \mid F(p) = 0 \} \) is a codim 1 linear subspace of \( \mathbf{V}_d \).

**Proof:** Any \( p = [a, b, c] \). Let \( F = \sum_{i+j+k=d} t_{ijk} x^i y^j z^k \). Then \( F(p) = 0 \) iff \( \sum_{i+j+k=d} t_{ijk} a^i b^j c^k = 0 \). This is a linear condition on the coefficients. \( \Box \)

**Cor:** Given any \( p_5, p_6, \ldots, p_r \in \mathbb{P}^2 \),
\[ \mathbf{V}_d (p_5, \ldots, p_r) = \{ F \mid F(p_i) = 0 \text{ for all } 5 \leq i \leq r \} \]
is a linear space of codim \( r-2 \).

**Example:** \( r \) can happen that codim \( < r \).
Take conics through \( p_1, \ldots, p_r \in \mathbb{P}^2 \).
- \( \emptyset \), the \( p_i \) are noncollinear, then we get a codim 4 subspace.
- \( \emptyset \), the \( p_i \) are collinear, then any conic through \( p_1, p_2, p_3 \) contains the line \( l \) which \( p_4 \) lies on, so remains.
Riemann's Thm: $\mathbb{X}$ = smooth proj. curve of genus $g$, $D$ = divisors on $\mathbb{X}$ of degree $d$, then
\[ \dim L(D) \geq d + 1 - g. \]

Simplifying assumption:
- $D$ effective, $D = p_1 + \cdots + p_r$, $p_i$ are distinct

Proof: (1) Realize $\mathbb{X}$ as a plane curve with only nodes, i.e., consider
\[ \phi: \mathbb{X} \to \mathbb{X}', \text{ birational morphism, } \mathbb{X}' = \text{plane curve} \]
\[ \text{of degree } f (\mathbb{X}' = \{ F = 0 \}) \text{ with only ordinary double point, } \]
\[ \Delta = \text{nodes of } \mathbb{X}', \# \Delta = s \]
Assume that more of the $p_i$ are in $\Delta$.
Recall $g = \frac{(f-1)(f-2)}{2} - s$.
Fix curve $B = B_0$ of degree $\geq 0$, not vanishing on $\mathbb{X}$ s.t.
(1) $B = 0$ passes through $p_1, \ldots, p_d$ and $\Delta$

(2) $B$ meets $\mathbb{X}$ in other pts as well: $R$ set of all such wed points.

How many pts in $R$?
By Bezout, $f \cdot c = \deg (\mathbb{X} \cdot B)$
\[
\text{As } \deg R = \delta e - 2\delta - d. \quad (\text{Assume for simplicity that } R \text{ consists indicated number of distinct } \mathfrak{p}s)
\]

(3°) Now let \( V(R, \Delta) = \sum \text{homo polys } A \) of degree \( e \) vanishing on \( R \) and \( \Delta \).

Here a homomorphism

\[
\begin{align*}
V(R, \Delta) & \overset{\rho}{\longrightarrow} \mathcal{Z}(D) \\
A & \longmapsto \varphi^*(A_{B_0})
\end{align*}
\]

Then we have:

\[
\left( e \right) - \deg R - \delta \leq \dim V(R, \Delta)
\]

i.e., \( \dim V(R, \Delta) \geq \left( e \right) - \delta + d \).

(4°) \( \rho \) is not injective.

In fact, \( \rho(A) = 0 \) iff \( \varphi^*(A_{B_0}) = 0 \)

iff \( \text{FA} \).

Kernel(\( \rho \)) = \( \sum A \mid A = FA \) s.t. \( \deg A = e - f_2 \).

\[
\Rightarrow V_{e,f} = \sum A_2's \sum f_2's.
\]

i.e.,

\[
0 \rightarrow V_{e,f} \overset{-F}{\longrightarrow} V(R, \Delta) \overset{\rho}{\longrightarrow} \mathcal{Z}(D)
\]

Then \( \dim \mathcal{Z}(D) \geq \dim V(R, D) - \dim V_{e,f} \)

\[
\geq \left( e \right) - \delta e + \delta + d - \left( e \right) f_2
\]

\[
= \frac{1}{2} \left( (e + 2)(e + 1) - \delta f_2 \delta + \delta d - (e - f_2)(e - f_2 + 1) \right)
\]

\[
= d + 1 - g. \quad \text{Resulting formula for } g.
\]
Riemann-Roch: \( \mathbb{X} = \text{proj curve, genus } g \), \( D \) any divisor on \( \mathbb{X} \).

\[ K = \text{div}(w) \text{ for } w \text{ a mer 1-form}, \quad \mathcal{L}(D) = \text{div}(L(D)). \]

Then

\[ \mathcal{L}(D) = d + 1 - g + \ell(K-D). \quad (d = \text{deg } D) \]

Note: \( c_0 D \) is effective,

\[ \mathcal{L}(K-D) \cong \left\{ \text{holo 1-forms vanishing on } D \right\}. \]

Because

\[ \mathcal{L}(K-D) = \left\{ \text{holo 1-forms with } \text{div}(w) = 0 \right\}. \]

Note: \( \ell(K) = \dim \left\{ \text{holo 1-forms} \right\} \geq g. \)

Proof: Realize \( \mathbb{X} \) as a plane curve \( v \) modulo: \( \Phi: \mathbb{X} \to \mathbb{R}^2. \)

\[ \left\{ \text{holo 1-forms} \right\} \cong \left\{ \text{homog poly os degree } d-3 \text{ vanishing at } \Phi(v) \right\}. \]

\[ \Rightarrow \dim \geq \left( \frac{d-1}{2} \right) - 3 = g. \]

Main Lemma: Let \( D \) be an effective divisor of degree \( d \) on \( \mathbb{X} \). Then

\[ \ell(D) \leq d + 1 - g + \ell(K-D). \]

Proof: (1) Fix a basis \( w_1, \ldots, w_5 \in \left\{ \text{holo 1-forms on } \mathbb{X} \right\}. \quad (s \geq g) \]

Assume for simplicity that \( D = P_1 + \cdots + P_s \) distinct points.

Let \( f_1, \ldots, f_s \in \mathcal{L}(P_1, \ldots, P_s) \) be a basis.

The residue theorem says for each \( i = 1, \ldots, s \),

\[ \sum_j \text{Res}_{P_i} (f_j w_j) = 0. \]

This puts conditions on the \( f_i \) and \( w_i \) need to come handy.

(2) Choose local coordinates \( z_i \) at \( P_i \). \( (z_j = 0 \text{ corresponds to } P_j) \).

Write \( w_j = \phi_i(z_j) \partial z_j \text{ (mean } P_j) \)

\[ f \partial z_i = \frac{p_i}{z_i} + \text{holomorphic in } z_j \text{ (mean } P_j). \]
\[ \text{Res}_j p_j (w_i) = \beta^j_i w_i(p_j) \]

For \( 1 \leq i \leq K \):

\[ \vec{V}_k = \begin{pmatrix} \beta_{1k} \\ \vdots \\ \beta_{Kk} \end{pmatrix} \]

"vector of principal parts of \( f \)."

1. Consider the matrix:

\[
\begin{bmatrix}
    w_1(p_1) & \cdots & w_1(p_d) \\
    \vdots & \ddots & \vdots \\
    w_s(p_1) & \cdots & w_s(p_d)
\end{bmatrix}
= M
\]

"Brill–Noether matrix."

For each \( c_k \), \( \sum_j \text{res}_j p_j (f_w w_j) = 0 \). \( \forall i \). \( \implies \)

\[ M \cdot \vec{v}_k = 0 \]

Note: \( \vec{v}_k = 0 \iff \text{for \ a \ haw \ no \ poles} \iff \text{for \ a \ constant} \).

So \( \ell(P_1, \ldots, P_d) \leq 1 + \dim \ker M \)

\[ \begin{bmatrix}
    \ell & \text{K constants}
\end{bmatrix} \]

2. What is \( \dim \ker M \)?

If \( M \) has full rank:

\[ \dim \ker M = (d-s) + \# \text{linear relations among rows of } M \]

A linear relation among the rows of \( M \) is the same as giving a hole \( w \) vanishing at \( P_1, \ldots, P_d \).

\[ \# \text{linear relations among rows} = \ell(K-D) \implies \]

\[ \dim \ker M = d-s + \ell(K-D) \]

So \( \ell(P_1, \ldots, P_d) \leq 1 + d-s + \ell(K-D) \leq 1 + d-g + \ell(K-D) \).
Proof: The proof of the theorem shows:

\[ d + 1 - g \leq \deg(P_1 + \cdots + P_d) \leq 1 + d - s + \deg(K - D). \]

Riemann's thm.

Now take \( d > 2g - 2 \), then \( \deg(K - D) < 0 \Rightarrow \deg(K - D) = 0 \)

\[ \Rightarrow \; d + 1 - g < 1 + d - s \quad \text{i.e.}, \quad s > g. \quad \text{But we already know} \; s \geq g. \]

Lemma: Let \( D \) be an effective divisor. Then RR holds for \( D \), i.e., \( \ell(D) = d + 1 - g + \deg(K - D) \).

Proof:

\textbf{Case 1:} \( \deg(K - D) = 0 \)

Then we are done by Main lemma and Riemann's thm.

\textbf{Case 2:} \( \deg(K - D) \neq 0 \).

As \( \deg(K - D) \neq 0 \), take effective \( E = K - D \).

Apply the main lemma to \( E \):

\[ \ell(E) \leq \deg(E) + 1 - g + \deg(D) \]

\[ = (2g - 2 - d) + 1 - g + \deg(D) \]

\[ \Rightarrow \; \ell(D) \geq d + 1 - g + \deg(K - D). \quad \text{Using} \; \ell(E) = \ell(K - D). \]
**Remark:** RR holds for Diff it holds for K-D.

**Proof (RR):** Need: \( l(D) = d + 1 - g + l(K-D) \).

- \( l(D) > 0 \Rightarrow D \geq \text{effective divisor}, \text{previous lemma applies.} \)
- \( l(K-D) > 0 \Rightarrow \text{previous lemma applies to K-D, remark gives proof.} \)

So it only remains to prove \( l(D) = l(K-D) = 0 \).

By Riemann's inequality,

\[
\begin{align*}
l(D) &\geq d + 1 - g \\
l(D) = 0 &\Rightarrow d \leq g - 1. \\
l(K-D) &\geq 2g - 2 - d + 1 - g \\
&= g - 1 - d \\
l(K-D) = 0 &\Rightarrow d > g - 1.
\end{align*}
\]

Thus \( d = g - 1 \).

But then this is exactly what RR says. \( \blacksquare \)
Def: The index of speciality of $D$ is $l(K-D)$; $i(D)$.

Claim: If $d \geq 2g-1$, then $i(D) = 0$. So $l(D) = d+1-g$.

Example: $g=1$. If $d \geq 1$, then $l(D) = V_d = d$.
If $d = 0$, $l(D) = 0$ if $D \neq 0$, $l(D) = 1$ if $D = 0$.

Proof: If $d \geq 2g-1$, then $\deg(K-D) = (2g-2) - d < 0$. Thus $l(K-D) = 0$.

Morphisms to $\mathbb{P}^r$, Linear Series:

$|D| =$ complete linear series associated to $D$.

\[ d \rightarrow \text{div}(h) \rightarrow \mathbb{P}(\mathcal{L}(D)) \]
\[ \rightarrow \mathcal{D}_{\text{eff}} \] \[ \rightarrow \mathbb{P}(\mathcal{O}(D)) \] \[ \rightarrow \mathbb{P}(\mathcal{O}(D)) \]

$|D| = \mathbb{P}(\mathcal{O}(D))$. By

$r(D) =$ dim $|D|$ is the dimension of $\mathbb{P}(\mathcal{O}(D))$.

$r(D) = l(D) - 2$.

Any $r(D) = r \geq 1$.

We want to find a morphism $\phi = \phi_{|D|} : X \rightarrow \mathbb{P}^r$, $\phi(x) \neq$ any hyperplane.

$|D| = \{ x + H \mid H \in \mathbb{P}^r \text{ hyperplane} \}$. 

\[ X \rightarrow \mathbb{P}^r. \]
Def: \( ID_1 \) (or \( D_1 \)) is base point free (or free) if given any \( P \in \mathbb{P} \), \( \exists D \in ID_1 \) s.t. \( \text{ord}_P(D) = 0 \). Def. 

Prop: \( D \) is free, then \( \exists \phi_{ID_1}: \mathbb{P} \to \mathbb{P}^r \) with 
\[
\text{ID}_1 = \mathbb{P} \times_{\mathbb{P}^n} \mathbb{P}.
\]

(Choose a basis \( 1, f_1, \ldots, f_r \in L(D) \). Define \( \phi_D = \langle f_1, f_2, \ldots, f_r \rangle \)

More intrinsically, assume \( \text{ID}_1 \) free. \( \mathbb{P}^r = \text{ID}_1^* \) a hyperplane in \( \text{ID}_1 \).

\[
\phi_{ID_1}: \mathbb{P} \to \mathbb{P}^r
\]

\[
\phi_{ID_1}(P) = \langle D' \mid \text{ord}_P(D') \geq 1 \rangle
\]

Lemma: \( \text{ID}_1 \) is free iff \( \chi(D-P) = \chi(D) - 1 \) \( \forall P \in \mathbb{P} \).

On this... \( \text{ID}_1 - P_1 \to \text{ID}_1 \). The image is \( \mathbb{P}^r \mid \text{ord}_P(D) \geq 1 \).

Lemma: Assume \( \text{ID}_1 \) is free, so we have \( \phi_{ID_1}: \mathbb{P} \to \mathbb{P}^r \).

\( \phi_{ID_1} \) is an embedding iff \( \chi(D-P-Q) = \chi(D) - 2 \) \( \forall P, Q \in \mathbb{P} \).

Idea of proof:

\( \text{ID}_1 - P - Q \to \text{ID}_1 \)

\( E \to \text{E} + P + Q \).

Assume \( P \neq Q \).

\[ P \xrightarrow{Q} P \]
Given $P, Q$,

$1D - P - Q \cong 1D$

is the set of all hyperplane sections passing through $P, Q$.

$C$ fails to be an embedding iff $C$ not 1-1, $dC = 0$ at some pt.

Suppose $C$ is not 1-1. Then $\exists P, Q$ s.t. $C_p = C_Q$.

Any hyperplane $H$ through $O$ has $S \cdot H$ contains $P, Q$.

As $1D - P - Q \cong 1D$ has codim 2. \[\varepsilon (1D - (2 - D) = 1D - D - 1).\]

Recap: $\forall \epsilon 1D - P - Q \cong 1D - 2 \forall P, Q$, then $C_{1D}$ is an embedding.

Case 2. $D$ a divisor on $X$.

- $\forall \epsilon dD = \epsilon g$, then $1D_{1D}$ is free.
- $\forall \epsilon dD = \epsilon g + 1$, then $C_{1D}$ is an embedding. (is very ample)

(e.g. particular, taking $\epsilon = g + 1$, get an embedding)

$X \to P^{g+1}$ or $\deg d$
Proof: Any \( \deg D \geq g+1 \). \( \forall P, Q, \ deg(D-P-Q) \geq g-1 \),
so \( \ell(D-P-Q) = 0 \).

RR: \( \ell(D) = d+1-g = \ell(D-P-Q) = (d-2) + 1-g \)

\[ = d-1-g \]
\[ = \ell(D) - 2 \]

As we are getting embeddings of these curves into large projective spaces.

Example: Any \( \mathbb{X} \) of genus \( g \) can be expressed as \( \mathbb{X} \to \mathbb{P}^1 \)
of \( \deg g+1 \).

We'll come back to this later.

Most Interesting Case: Apply this to \( \mathbb{X}_{\text{can}} \).

\( \ell(K) = \dim \mathbb{X}_{\text{can}} \) holomorphic 1-forms \( \mathbb{X}_{\text{can}} = g \).

\( \dim \mathbb{K} = g-1 \).

\( \mathbb{C} \mathbb{P}_{K}: \mathbb{X} \to \mathbb{P}^{g-1} \). "Canonical map"
Recall:

$X$ = smooth proj. curve genus $g$

$D = \text{effective divisor on } X, \text{ deg } D = 2g - 2$

$|D| = \{ \mathcal{D} \mid \mathcal{D} \sim D \}$

Prop:

1. Assume $P \in X$, $l(D-P) = l(D) - 3$. Then $\exists \phi_P : X \to \mathbb{P}^{n}$

   such that $l(D) = \int X \cdot H^2$, $H \in \mathbb{P}^n$ hyperplane.

2. $\forall \ell \geq l(D-P) = l(D) - 2$, $P, Q \in X$, then $\phi_P$ is an embedding.

Canonical Mapping:

$D = K - \text{div}(w)$, $w = \text{hol} i - \text{form}.$

Assume that $g \geq 2$.

Lemma: $|K|$ is base point free, i.e., get $\phi_{|K|} : X \to \mathbb{P}^{g-1}$, $l(K) = g$.

Proof:

Let $P \in X$ such that $l(K-P) = l(K) = g$. Apply R.R to $K-P$.

$l(K-P) = \deg(K-P) + 1 - g + l(K-(K-P))$

$l(K-P) = (2g - 3) + 1 - g + l(P)$.

Since $g = g - 2 + l(P)$

i.e., $\exists P$ s.t. $l(P) = 2$.

i.e., $\dim l(P) = 2$.

$\Rightarrow \exists \text{ a meromorphic } f \in l(P)$

We can view $f$ as defining $\tilde{f} : X \to \mathbb{P}^1$.

$\tilde{f}(\infty) = P$ since only a single pole

$\Rightarrow \deg \tilde{f} - 1 = f$ isomorphism.

# $g \geq 3$. \end{proof}
**Alternative Interpretation:**

Def: \( H^1(\mathcal{E}) = \frac{3}{2} \) hole 1-forms 3.

Choose a basis \( \omega_1, \ldots, \omega_3 \in H^1(\mathcal{E}) \).

\( \phi_{1k}: \mathcal{E} \rightarrow \mathbb{P}^3 \)

\( x \mapsto [\omega_1(x), \ldots, \omega_3(x)] \)

"Canonical mapping"

We have said before that \( \mathcal{L}(K) \cong H^1(\mathcal{E}) \).

**Thm:** The canonical mapping is an embedding unless \( \Sigma \) is hyperelliptic, i.e., \( \Sigma \) admits a 2-1 branched covering \( \tau: \Sigma \rightarrow \mathbb{P}^1 \) of degree 2.

**Proof:** Assume \( \phi_{1k} \) is not an embedding. Let \( \Sigma, \mathcal{L}, \mathcal{P} \) s.t.

\( \mathcal{L}(K-P-Q) > \mathcal{L}(K-C) - 2 = g-3 \).

Then \( \mathcal{L}(K-P-Q) = g-1 \) by previous lemma.

Apply RR:

\( g-1 = \mathcal{L}(K-P-Q) = (2g-2-2) + 1 - g + \mathcal{L}(P+Q) \)

\( \Rightarrow \mathcal{L}(P+Q) = 2 \). (since \( \mathcal{L}(P+Q) = 2 \).

So as before, we get a meromorphic map \( \mathcal{S} \in \mathcal{L}(P+Q) \),

so gives a 2-1 map \( \Sigma \rightarrow \mathbb{P}^1 \).

**Hyperelliptic Curves:** (93.2)

\( \Sigma \) genus \( g \), admits \( \Sigma \rightarrow \mathbb{P}^1 \) of degree 2.

\( \Sigma \) is the R.S. associated to \( y^2 = f(x) \).

distinct roots
The number of ramification points $P$ of $\pi$:

$$(\deg - 1) + 2g + 2 = \# \text{ ramification points}. $$

$$\deg \pi = - \log(1/2).$$

$$= 2g + 2. \quad (R1)$$

$$y^2 = \begin{cases} f_{2g+1} & \text{one ramification at } \infty. \\ f_{2g+2} & \end{cases}$$

Note:
- Have described these hyperelliptic curves completely explicitly.
- In large genus, hyperelliptic curves are completely atypical among all curves of genus $g$.

Claim: Every curve of genus $g + 2$ is hyperelliptic.

Proof: Canonical map is $X \rightarrow \mathbb{P}^1$. 

$$\deg \phi = \deg \psi = 2g + 2 = 2 \cdot 3.$$ 

Alternately, $L(\mathcal{K}) = 2g + 2 = 3$ nonet $f \in L(\mathcal{K})$. We can write $f : X \rightarrow \mathbb{P}^1$. Then $\deg \mathcal{K} = 2$, 

$$f(x) = x^{2g+2}.$$ 

Note: $(g - 2)$

$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ 

The canonical map $X \rightarrow \mathbb{P}^1$ ramifies at 5 points. These 5 points are intrinsically defined special points on $X$.

These are the Weierstrass points.
Genus 3:

**Theorem:** Let \( X \) be a non-hyperelliptic curve (n.h.) of genus 3, then \( X \) can be realized as a smooth quartic in \( \mathbb{P}^2 \). Conversely, any smooth quartic is the canonical embedding of a curve of genus 3.

**Proof:** Canonical embedding: \( X \to \mathbb{P}^{g-1} = \mathbb{P}^3 \), and \( \deg X = 2g-2 = 4 \).

So we get a plane quartic.

Conversely, any \( X \subset \mathbb{P}^2 \) is a smooth quartic. Then genus of \( X \) is \( (7^3) = 3 \).

Recall: \( Y \subset \mathbb{P}^2 \) is curve of degree \( e \), then \( \deg \text{can}(X) = 3 \cdot \deg(Y) \).

This shows that in genus 3,

\[ X \subset \mathbb{P}^3 \text{ a plane quartic} \]

\[ K = X \cdot H \]

So \( X \) is embedded by the canonical linear series.

\[ X = \text{non h.o. curve of genus 3; } (X \subset \mathbb{P}^3 \text{ of deg 4}). \]

**Question:** What is the least degree of a branched covering \( X \to \mathbb{P}^2 \)?

It must be at least 3 because it isn’t hyperelliptic. Can we do it for 3? Can we do it for 4 just by projecting? To get degree 3, project from a point on the curve, which is a 3-1 map.
So we get \( \pi: \mathbb{X} \to \mathbb{P}^1 \) of degree 3 by projection from any point \( P \in \mathbb{X} \subseteq \mathbb{P}^2 \).

\[ I(K-P) : \]
\[ I(K-P) = I(K) - 1 = 2 \quad (K \text{ is base pt. free}) \]

\[ \text{Note: } I \text{ a 1-dim family of degree 3 curves } \mathbb{X} \to \mathbb{P}^1. \text{ (Every point on the curve gives this)} \]
Recall: $\mathcal{X}$ curve of genus $g \geq 2$.
Choice basis $v_1, \ldots, v_g \in H^1(\mathcal{X})$.

Define $\varphi_{1k_1}: \mathcal{X} \to \mathbb{P}^{g-1}$
\[ x \mapsto \{v_1(x), \ldots, v_{g}(x)\}. \]

**Thm:** This is an everywhere defined morphism, i.e., not all the $1$-forms vanish at any point. i.e., $\mathcal{X}$ is base point free.
$\varphi_{1k_1}$ is an embedding unless $\mathcal{X}$ is hyperelliptic, i.e., if $\mathcal{X} \to \mathbb{P}^1$ deg $2$.

**Example:** $g = 3$. Either $\mathcal{X}$ is hyperelliptic, or $\mathcal{X} \to \mathbb{P}^2$ is a smooth plane quartic.

**Genus 4:**

**Thm:** $\mathcal{X}$ a non-hyperelliptic curve of genus 4. Then $\mathcal{X}$ is a complete intersection of a quadric and cubic surface in $\mathbb{P}^3$. (Conversely, any such complete intersection is a canonical curve of genus 4.

Complete intersection = transversal intersection of 2 surfaces.

**Proof:** Look at canonical embedding:
\[ \mathcal{X} \to \mathbb{P}^3 \quad \text{by} \quad 1, f_1, \ldots, f_3, L(K) \quad x \mapsto \{1, f_1(x), f_2(x), f_3(x)\} \]

$\mathcal{X}$ has degree $6 = 2g - 2$.

Let $V_p = \{ \text{homog poly of deg p on } \mathbb{P}^3 \}$. $\dim V_p = \binom{p+3}{3}$

Then we have:
\[ p_p: V_p \to L(pK) \]
\[ \psi \quad \psi \]
\[ F(x_0, \ldots, x_3) \to F(1, f_1, \ldots, f_3) \]

$\ker p_p = \mathcal{I}_p = \{ \text{homog poly of deg } p \text{ vanishing on } \mathcal{X} \}$. 

\textbf{Claim:} \( \mathcal{X} \) lies on a unique (up to scalars) quadric surface \( F \).

\[ \beta : \mathcal{V}_3 \to \mathcal{L}(2k) \]

\[ \dim 10 \quad \mathcal{L}(2k) = \deg 2k + 1 - g + 0 \]
\[ = 12 + 1 - 4 \]
\[ = 9 \]

As there is at least a 1-dim kernel.

\textbf{Remark:} \( \beta \) is an iso by construction.

\textbf{Claim:} \( \dim \ker \beta = 1 \)

\textbf{Pf:} Suppose to contrary that \( Q_1, Q_2 \in \ker \beta \). These have no common factors or else \( \beta \) would not be an iso.

Take a hyperplane \( H \).

\[ \begin{array}{c}
\text{H} \\
\# \mathcal{X} \cap H = 6
\end{array} \]

But we can't have two conics intersecting in 6 points.

So \( \deg Q \cap Q \leq 4 \), so \( \mathcal{X} \notin Q_1 \cap Q_2 \)

Thus the claim is satisfied.

\textbf{Cubic:}

\[ \beta : \mathcal{V}_3 \to \mathcal{L}(3k) \]

\[ \dim 0 \quad \dim 15 \]

So now we have a kernel of dimension at least 5.

\( \ker \beta \ni X_0 F, X_1 F, X_2 F, X_3 F \).

By dimension, \( \exists \) cubic \( G \in \ker \beta \) s.t. \( G \) and \( F \) have no common factors.
\[ X \subseteq F \cap G \quad (F = \omega, \, \lambda G = \omega_1). \]

\[ \sum_{\text{dyke}} \quad \text{dyke by Bezout.} \]

As there is nothing left, \( X = F \cap G. \)

**Question:** Does \( X \) admit a 3-1 map to \( \mathbb{P}^1 \)?

\[ F = F_c \quad \text{can. set by } c \]

\[ \text{rk} F = \begin{cases} 4 & \text{Famous} \\ 3 & \text{F cone over plane conic} \end{cases} \]

The cubic is not intrinsic, but the quadric is.

\[ F, \text{ a non-sing quadric (rk4).} : \]

\[ F \cong \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{(Segre embedding).} \]

\[ \begin{array}{c}
F \\
\text{each line meets the curve} \\
\text{three times counting mult.}
\end{array} \]

In this case, \( X \) admits 2 3-1 maps onto \( \mathbb{P}^1 \),

one for each projection.

\[ F, \text{ cone over plane conic:} \]

Projecting gives a 3-1 map to \( \mathbb{P}^1 \), but in

this case this is the only one.
Are any two non-hypereflect curves of \( g = 3 \) isomorphic?

\[ \Sigma \in \mathbb{P}^2 \] is a plane quartic (defined up to change of coordinates)

Plane quartics = \( \mathbb{P}^{14} \) = \( \bigcup \) non-\( \Sigma \) curves

\( \text{PGL}(3) = G \) = change of coordinate

has \( \dim \, G \).

This says they aren't all isomorphic.
9.1: Curves are all elliptic. Use Riemann-Roch.

9.2: Curves are hyperelliptic. Use canonical map $\phi_{1k}: \mathbb{B} \to \mathbb{P}^{3}: \mathbb{P}^1$ which has degree $\text{deg} \ k = 2g - 2 = 2$.

9.3: If $\mathbb{B}$ is not hyperelliptic, then it is a smooth quartic in $\mathbb{P}^3$ and vice versa. Use the canonical map again to get $\phi_{2k}: \mathbb{B} \to \mathbb{P}^{3}: \mathbb{P}^2$ of degree $2g - 2 = 4$.

9.4: If $\mathbb{B}$ is not hyperelliptic, then it is a complete intersection (i.e., transversal intersection) of a quadric and cubic surface in $\mathbb{P}^3$ and vice versa.
Jacobian:

$\Sigma$ = smooth projective curve (compact $\mathbb{P}^g$ of genus $g$)

Recall:

$H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$

Beauis homology group.

$H_1(\Sigma; \mathbb{C}) \cong \mathbb{C}^{2g}$

$H^0_{dr}(\Sigma; \mathbb{C}) = H^0(\Sigma)$

\[
\begin{align*}
\text{closed} & \subset \mathcal{C}^0 \text{ 1-forms} \\
\eta \in \text{closed} & \subset \text{exact forms} \\
\eta & \in \mathcal{C} \text{ exact forms}
\end{align*}
\]

Recall:

1. The bilinear map

\[
H^0_{dr}(\Sigma; \mathbb{C}) \otimes H_1(\Sigma; \mathbb{C}) \longrightarrow \mathbb{C}
\]

$$(\eta, \gamma) \longrightarrow \int_\gamma \eta$$

is a perfect pairing.

2. The (alternating) pairing

\[
H'(\Sigma) \otimes H'(\Sigma) \longrightarrow \mathbb{C}
\]

$$(\alpha, \beta) \longrightarrow \int_\Sigma \alpha \wedge \beta$$

is non-degenerate. This is a form of Poincaré duality.
$H^{1,0}(\mathbb{X}) = \left\{ \text{holo } 1\text{-forms } \right\}$ is a complex vector space of complex dimension $g$.

We showed before a holo $1$-form is closed, so we get:

\[ H^{1,0} \rightarrow H^1(\mathbb{X}). \]

**Def.** An anti-holomorphic $1$-form is conjugate to a holo $1$-form, i.e., any $\omega \in H^{1,0}(\mathbb{X})$. Locally $\omega = f(z)dz = f(z)(dx + idy)$, \[ \text{holo, then } \overline{\omega} = f(\overline{z})d\overline{z} = f(\overline{z})(d\overline{x} - id\overline{y}). \]

\[ H^{0,1}(\mathbb{X}) = \left\{ \text{vector space of all anti-holo } 1\text{-forms } \right\} \]

\[ \dim_{\mathbb{C}} H^{0,1} = g. \]

**Exercise:** If $\omega$ is holo, then $d\overline{\omega} = 0$, i.e., anti-holo forms are closed.

So we have $2$ maps:

\[ H^{1,0}(\mathbb{X}) \rightarrow H^1(\mathbb{X}) \]

\[ H^{0,1}(\mathbb{X}) \rightarrow H^0(\mathbb{X}) \]

**Then:** These maps are injective, and we have a direct sum decomposition:

\[ H^1(\mathbb{X}, \mathbb{C}) = H^{1,0}(\mathbb{X}) \oplus H^{0,1}(\mathbb{X}). \]

More concretely, choose a basis $w_1, \ldots, w_g$ of $H^{1,0}(\mathbb{X})$, then $\overline{w}_1, \ldots, \overline{w}_g$, $\overline{w}_1, \ldots, \overline{w}_g$ are a basis of $H^1(\mathbb{X}, \mathbb{C})$. 
Remark: This is a "baby" case of the Hodge decomposition.

Proof: It is enough to show that the cup product pairing among $w_i$, $\bar{w}_j$ is nondegenerate. This will give the linear independence, which we claim. Counting gives the thm.

For reasons of type

$$\int_X w_i \wedge \bar{w}_j = 0 \quad \text{all } i,j$$

$$\int_X \bar{w}_i \wedge \bar{w}_j = 0$$

In fact, $w_i \wedge \bar{w}_j = 0$, $\bar{w}_i \wedge \bar{w}_j = 0$ because $w_i = f \zeta_i dz$, $\bar{w}_j = g \zeta_{-j} dz$, and $w_i \wedge \bar{w}_j = f \cdot g \zeta_i \zeta_{-j} d\zeta d\bar{\zeta} = 0$

Let $a_{ij} = \int_X w_i \wedge \bar{w}_j$.

Claim: If $A = (a_{ij})$, then $iA$ is a positive definite Hermitian matrix.

Granting the claim, it follows that the matrix for cup product in terms of $w_1, \ldots, w_j, \bar{w}_1, \ldots, \bar{w}_j$, is

$$\begin{pmatrix} 0 & A \\ -\bar{A} & 0 \end{pmatrix}$$

i.e. $iA > 0$.

And this is non-degenerate.

Claim: Fix $0 \neq \omega \in H^{1,0}(\mathbb{C})$. Need to show

$$\int_X \omega \wedge \bar{\omega} > 0.$$ In local coordinates, $\omega = f \zeta \bar{\zeta} d\zeta$, $\bar{\omega} = \bar{f} \zeta \bar{\zeta} d\zeta$

$\bar{\omega} = \bar{f} \zeta \bar{\zeta} (d\zeta - d\bar{\zeta}) \Rightarrow \omega \wedge \bar{\omega} = i f \zeta (\bar{f} \zeta)^2 (d\zeta - d\bar{\zeta})$

Thus, $\int_X \omega \wedge \bar{\omega} = 2 \int_{\mathbb{D}} f^2 (dx \wedge dy) > 0$. □
As \( H^1_{\text{tr}}(\Sigma) \oplus H^0(\Sigma) \subset H^1(\Sigma) \), since \( \dim H^1(\Sigma) = 3 \), we get the result.

**Proof:** Pairing \( \langle \omega, \eta \rangle = \int_{\Sigma} \omega \cdot \eta \) is a pos. def. Hermitian form on \( H^{1,0}(\Sigma) \).

\[ H^{1,0}(\Sigma) \subset H^1(\Sigma). \]

\[ C^9 \quad C^{2g} \]

Now define a map:

\[ \gamma : H^1(\Sigma, \mathbb{Z}) \rightarrow H^{1,0}(\Sigma)^* \]

\[ \gamma \rightarrow T_\gamma = \int_Y \]

Where \( T_\gamma(\omega) = \int_Y \omega \)

**Note:** This is the comp. \( H^1(\Sigma, \mathbb{Z}) \rightarrow H^1_{\text{tr}}(\Sigma, \mathbb{C})^* \)

\[ 2 : \mathbb{Z}^{2g} \rightarrow C^9 \]
**Thm:** The homomorphism $i$ is injective and $\Lambda = \text{im}(i) \subseteq H^0(\Sigma)^*$ is a lattice, i.e., $i$ takes a basis of $H_1(\Sigma, \mathbb{Z})$ in $\mathbb{R}^N$ linearly independent vectors in $H^0(\Sigma)^*$.

\[ H^0(\Sigma)^* \]

(like in 1-dim case with elliptic curves)

i.e., $\mathbb{Z}^g \cong \Lambda \subseteq H^0(\Sigma)^* \cong \mathbb{C}^g$

**Def:** The Jacobian of $\Sigma$ is $\text{Jac}(\Sigma) = \frac{H^0(\Sigma)^*}{\Lambda}$, a complex torus of dim $g$.

**Proof (Thm):** Let $\gamma_1, \ldots, \gamma_g \in H_1(\Sigma, \mathbb{Z})$. Suppose $\Sigma$

\[ \lambda_1, \ldots, \lambda_g \in \mathbb{R} \text{ s.t. } \sum \lambda_i \gamma_i = 0 \text{ in } H^0(\Sigma)^*. \]

i.e., \((*)\) \[ \sum \lambda_i \int_{\gamma_i} \omega = 0 \text{ for all } \omega \in H^0(\Sigma). \]

$\omega \in H^0(\Sigma)$. Since the $\lambda_i$ are real,

\[ (*) \] \[ \sum \lambda_i \int_{\gamma_i} \bar{\omega} = 0 \text{ for all } \omega \in H^0(\Sigma). \]

Using $\int_{\gamma_i} \omega = \int_{\gamma_i} \bar{\omega}$. But since $H^0 \otimes H^0 = H^0$,

this implies $\sum \lambda_i \int_{\gamma_i} \eta = 0$ for all $\eta \in H^0(\Sigma)$.

Now by the dolbeau thm, $\sum \lambda_i \gamma_i = 0$ in $H_1(\Sigma, \mathbb{C})$. \(\square\)
Recall:

\[ \mathbb{E} = \text{projective curve of genus } g \]

\[ H_0^\omega(\mathbb{E}) = \text{holo } 1\text{-forms on } \mathbb{E} = \mathbb{C}^g. \]

\[ \iota: H_1(\mathbb{E}, \mathbb{Z}) \to H_1^\omega(\mathbb{E}) \]

\[ \gamma \mapsto \oint_{\gamma} \]

\[ H_1(\mathbb{E}, \mathbb{Z}) = \mathbb{Z}^g \]

\[ \Lambda = \text{im}(\iota) \text{ is a lattice (period lattice)} \]

**Definition:**

\[ \text{Jac}(\mathbb{E}) = \frac{H_1^\omega(\mathbb{E})^*}{\Lambda} \]

**Local coordinates:**

\[ \omega_1, \ldots, \omega_g \in H_1^\omega(\mathbb{E}) \]

\[ \gamma_1, \ldots, \gamma_{2g} \in H_1(\mathbb{E}, \mathbb{Z}) \]

\[ H_1(\mathbb{E}, \mathbb{Z}) \to H_1^\omega(\mathbb{E})^* \]

\[ \gamma \mapsto \oint_{\gamma} \omega = \left( \oint_{\gamma_1} \omega_1, \ldots, \oint_{\gamma_{2g}} \omega_g \right) \]

\[ \Lambda \text{ is generated by rows of a } g \times 2g \text{ matrix:} \]

\[
\begin{bmatrix}
\oint_{\gamma_1} \omega_1 & \ldots & \oint_{\gamma_1} \omega_g \\
\ldots & \ldots & \ldots \\
\oint_{\gamma_{2g}} \omega_1 & \ldots & \oint_{\gamma_{2g}} \omega_g
\end{bmatrix} = \text{period matrix} \quad \text{ob } \mathbb{E}
\]

\[ \text{Jac}(\mathbb{E}) \text{ is a complex torus of dimension } g, \text{ i.e., } \mathbb{C}^g/\Lambda \cong \mathbb{R}^g/\mathbb{Z}^g \]

\[ \cong S^1 \times \cdots \times S^1 \]

\[ \text{sm, copi} \]
Example: $\mathbb{X} = \mathbb{C}/\Lambda$.

$$\pi: \mathbb{C} \rightarrow \mathbb{C}/\Lambda = \mathbb{X}$$

$\pi^* H^{1,0}(\mathbb{X}) = \mathbb{C}:dz$,

$$dz = \pi^*(\omega).$$

The natural basis for $H_1(\mathbb{X}, \mathbb{Z})$ is $\alpha, \beta$ which are the images $\alpha, \tilde{z}$ and $\beta$.

$$H_1(\mathbb{X}, \mathbb{Z}) \rightarrow H^{1,0}(\mathbb{X})^*$$

$$(\alpha, \beta) \rightarrow \left( \int_{\alpha} \omega, \int_{\beta} \omega \right)$$

$$\int_{\alpha} \omega = \int_{\tilde{z}} d\tilde{z} = \tilde{z} \bigg|_{0}^{\lambda_1} = \lambda_1.$$  

$$\int_{\beta} \omega = \lambda_2.$$  

$$\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 \subseteq \mathbb{C}$$

So $\text{Jac} \left( \mathbb{C}/\Lambda \right) = \mathbb{C}/\Lambda$

### Abel-Jacobi Map:

Fix base point $p_0 \in \mathbb{X}$. We will define a holomorphic mapping

$$u: \mathbb{X} \rightarrow \text{Jac}(\mathbb{X}).$$
Fix $p \in X$.

Choose a path $\gamma_p$ from $p_0$ to $p$.

Given $w \in H^0(X)$, $\int_{\gamma_p} w \in C$, so get $H^0(X) \xrightarrow{\gamma_p} C$.

Note: $\int_{\gamma_p}$ depends on the path $\gamma_p$.

However, if $\gamma_p$, $\gamma'_p$ are two paths from $p_0$ to $p$, then $\gamma_p \cdot \gamma'_p$ is a 1-cycle on $X$.

So $\int_{\gamma_p} w - \int_{\gamma'_p} w = \int_{\gamma_p \cdot \gamma'_p} w$, i.e., thinking of these integrals as functionals, $\left\{ - \int_{\gamma_p} w \right\} \in \Lambda = \text{im}(\mathcal{L}) = \text{im}\left( H_1(X, \mathbb{Z}) \to H^0(X)^* \right)$.

Thus $\int_{\gamma_p} w \equiv \int_{\gamma'_p} w \pmod{\Lambda}$. i.e., $\int_{\gamma_p} w = \int_{\gamma'_p} w \pmod{\Lambda} \in \frac{H^0(X)^*}{\Lambda} = \text{Jac}(X)$.

So $u(p) = \int_p^1 \in \text{Jac}(X)$ is well-defined, independent of path, from $p_0$ to $p$.

$u : X \to \text{Jac}(X)$. 
Prop: \( U \) is a holomorphic map of complex manifolds.

Proof: Given \( p \), choose \( Y \) from \( p_0 \) to \( p \) and local coordinate \( z \) centered at \( p \). Choose \( w_1, \ldots, w_g \in H^1(\mathbb{C}) \) a basis, \( w_i = f_i e^{z^2} \) locally.

We need to show \( \int_{p_0}^{p} w_i \) vanishes in \( p' \) for \( p' \) near \( p \).

\[
\int_{p_0}^{p} w_i = \int_{p_0}^{p} w_i + \int_{p}^{p'} w_i
\]

This leads down to the statement that \( \int_{p} f(z) e^{z^2} \) is analytic in \( p' \), which we know from complex analysis.

We will see later that \( U : \mathbb{X} \to \text{Jac}(\mathbb{X}) \) is an embedding.

\[
\text{Hauso map: By means of translation on Jac(\mathbb{X})}, \text{ we have}
\]

\[
\text{a natural identification } T_p (\text{Jac}(\mathbb{X})) = T_0 \text{ Jac}(\mathbb{X}) \text{ (tangent space)}
= H^1(\mathbb{C})^*.
\]

So we get a \textit{Hauso map}.
\( Y : \mathbb{R} \rightarrow \mathbb{P} \mathcal{H}^0(\mathbb{R})^* \)

\[ \mathcal{P} \rightarrow T_p(\mathbb{R}) \leq T_0 \text{ Jac}(\mathbb{R}) \cong \mathcal{H}^0(\mathbb{R})^* \]

**Abel's Thm:** Given \( p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{R} \), there exists \( f \in C(\mathbb{R}) \)

with \( \text{div}(f) = \sum pi - \sum qi \) iff \( \sum u(p_i) = \sum u(q_i) \).

**Remarks:**

1. The definition of \( u \) depended on the base point \( P_0 \).

\[ U = U_{P_0} \]

This is ok though, because there is a canonical map \( \text{Div}^0(\mathbb{R}) \rightarrow \text{Jac}(\mathbb{R}) \)

\[ U(P-Q) = \int_{P_0}^P - \int_{P_0}^Q = \int_{P_0}^Q \]

As if we change the base point, the result changes so as to cancel it out.

2. \( \text{Div}^0(\mathbb{R}) \rightarrow \text{Jac}(\mathbb{R}) \), so Abel's thm is equivalent to saying \( \text{Jac}(\mathbb{R}) = \text{Div}^0(\mathbb{R})/\text{Prin}(\mathbb{R}) \)
$X$ is compact R.S. of genus $g$

Topologically $X$ is a $4g$-gon with sides identified in pairs.

As $X$ is:

\[
\Delta
\]

\[
\begin{align*}
\alpha & : j = 1, \ldots, g \\
\beta & : j = 1, \ldots, g \\
\end{align*}
\]

loops on surface corresponding to $\alpha_j$ and $\beta_j$

\[
\begin{cases}
\alpha_i \cdot \beta_i = 1 & \text{for } i \neq j \\
\alpha_i \cdot \alpha_i = \beta_i \cdot \beta_i = 0
\end{cases}
\]

Note: Let $\pi : \pi \rightarrow X$ be a universal cover of $X$ (as $\pi$ is a R.S.). We can view $\Delta \subset \pi$. So $\Delta \rightarrow X$ is holo.
Given finitely many points \( p_i \in X \), we can assume after translation that \( p_i \in \text{int}(\Delta) \).

Consider closed \( C^\infty \) (or holomorphic) 1-form \( \sigma \) on \( X \).

**Def:** \( A_i(\sigma) = \int_{a_i}^{b_i} \sigma \) and \( "a_i" \ and \ "b_i" \ periodic points of \( \sigma \) \( \int_{a_i}^{b_i} \phi \).

\( \sigma \) lifts to a form on \( \hat{X} \) and also on \( \hat{\Delta} \). Write \( \phi \)
for the form on \( \Delta \).

So \( \int_{a_i}^{a_i} \phi = \int_{a_i}^{a_i} \phi \) on \( \Delta \).

Fix \( p_0 \in \Delta \). Assume \( \sigma \) has no poles on \( \Delta \). For \( x \in \Delta \), define

\[
\int_{p_0}^{x} \phi = \int_{p_0}^{x} \phi \quad (\text{indep of path}: \phi \text{ closed, } \Delta \text{ simply connected})
\]

This is single valued on \( \Delta \) (or \( \Delta \) \( \backslash \) poles of \( \sigma \)).

\( \int_{p_0}^{p_0} \phi = \int_{p_0}^{x} \phi \) if \( \sigma \) is \( C^\infty \) (or holomorphic) if \( \sigma \) is \( \partial \sigma = 0 \).
Prop: Let $\sigma, \tau$ be $C^\infty$ closed 1-forms on $\Delta$. Then

$$\int_{\Delta} f_{\sigma \tau} = \sum_{i=1}^{g} \left( A_i(\sigma) B_i(\tau) - A_i(\tau) B_i(\sigma) \right)$$

Equality continues to hold if $\tau$ is nonzero with no poles on $\Delta$.

Proof: Choose $x \in a_i$ with corresponding point $x' \in a_i'$.

Claim: $f_{\sigma}(x) - f_{\sigma}(x') = -B_i(\sigma)$

Pf: Let $\alpha_x$ be a path from $x$ to $x'$. Then

$$f_{\sigma}(x) - f_{\sigma}(x') = \int_{\alpha_x} \sigma - \int_{\alpha_{x'}} \sigma$$

$$= \int_{x'}^{x} \sigma = -\int_{x'}^{x} \sigma.$$ But $\alpha_x \sim B_i, \alpha_{x'} \sim B_i(\sigma) \Box$.

Similarly, $y \in b_i, y' \in b_i'. \tau$ comes from a 1-form on $\Delta$, so it takes the same values on $a_i, a_i'$ (similarly on $b_i, b_i'$).

$$\int_{\Delta} f_{\sigma \tau} = \sum_{a_i} \left( \int_{a_i} - \int_{a_i'} + \int_{b_i} - \int_{b_i'} \right)$$

$$= \sum_{x \in a_i} \left( \int_{\sigma} \left( f_{\sigma}(x) - f_{\sigma}(x') \right) \tau \right) + \sum_{y \in b_i} \left( \int_{\tau} \left( f_{\tau}(y) - f_{\tau}(y') \right) \sigma \right)$$

$$= \sum_{a_i} \left( \int_{a_i} - B_i(\sigma) \tau \right) + \sum_{b_i} \left( \int_{b_i} A_i(\sigma) \tau \right)$$
Prop.: Suppose $w$ is any nonzero local 1-form on $X$.

Then $\text{Im} \left( \sum_{i=1}^{g} A_i(\omega) \overline{B_i(\omega)} \right) < 0$.

Proof: Recall $\int \omega \wedge \overline{\omega} = -2i \int (\text{strictly pos})$.

\[ \text{Im} \left( \int \omega \wedge \overline{\omega} \right) < 0 \]

Note: \[
\int_{\Delta} f \omega \cdot \overline{\omega} = \int_{\Delta} d(f \omega \cdot \overline{\omega}) = \int_{\Delta} df \omega \cdot \overline{\omega} + \int_{\Delta} f \omega \cdot d\overline{\omega}
\]

\[
= \int_{\Delta} \omega \wedge \overline{\omega} + i d\overline{\omega} = 0
\]

i.e., $\text{Im} \left( \int_{\Delta} f \omega \cdot \overline{\omega} \right) < 0$ \((\star)\)

Now we use the formula just derived:

\[
\int_{\Delta} f \omega \cdot \overline{\omega} = \sum_{i=1}^{g} (A_i(\omega) \overline{B_i(\omega)} - A_i(\overline{\omega}) \overline{B_i(\overline{\omega})})
\]

Also, $A_i(\overline{\omega}) = \overline{A_i(\omega)}$, $B_i(\overline{\omega}) = \overline{B_i(\omega)}$.

Thus, \[
\int_{\Delta} f \omega \cdot \overline{\omega} = 2i \text{Im} \left( \sum_{i=1}^{g} A_i(\omega) \overline{B_i(\overline{\omega})} \right) < 0.
\]

So by \((\star)\) $\text{Im} \left( \sum_{i=1}^{g} A_i(\omega) \overline{B_i(\overline{\omega})} \right) < 0$. 

Claim: If $\omega$ is a closed 1-form on $\mathbb{R}^n$, then

$$A_i(\omega) = 0 \forall i \iff \omega = 0 \quad \text{and} \quad B_i(\omega) = 0 \forall i \iff \omega = 0.$$
Period Relations:

\( X = \text{RS genus } g \).

View \( X \) as a \( 4g \)-gon with sides identified in pairs.

\[ \Delta \]

\( \phi \sigma \) is a closed \( C^0 \)-form on hole 1-form, view \( \sigma \) as a form on \( \Delta \). Fix \( p_0 \). Define \( \int_{\sigma} (x) = \int_{p_0}^{x} \sigma \), \( A_i(\sigma) = \int_{a_i} \sigma \),

\[ B_i(\sigma) = \int_{b_i} \sigma. \]

Then if \( \sigma, \tau \) are closed 1-forms,

\[ \int_{\sigma} \tau = \sum_{i=1}^{g} (A_i(\sigma) B_i(\tau) - A_i(\tau) B_i(\sigma)) \]

Prop: \( \phi \omega \) is any nonzero hole 1-form on \( X \), then

\[ \text{Im} \left( \frac{\partial}{\partial \omega_i} A_i(\omega) B_i(\omega) \right) < 0 \]

\[ \text{Im} \left( \frac{\partial}{\partial \omega_i} A_i(\omega) B_i(\omega) \right) > 0. \]

Corol: \( \phi \omega \) is a hole form and \( A_i(\omega) = 0 \) \( \forall i \), then \( \omega = 0 \). The same holds with \( B_i \) instead of \( A_i \).

Corol: Fix a basis \( \{ w_j \} \) of 1-forms \( H^1(\Delta) \). Then

the matrix \( A = (A_i(\omega_j)) = (\int_{a_i} \omega_j) \) is non-singular.

The same holds with \( A_i \) replaced by \( B_i \).
Proof: \[ A = \begin{bmatrix} \sum_{a_i} w_i \cdot a_j \end{bmatrix} \]

Now if \( A \) is singular, the rows are dependent.

Adding the rows gives

\[ \sum_{a_j} \lambda_i w_i = 0 \]

\[ \Rightarrow \sum_{a_i} \lambda_i w_i = 0 \quad \# \text{ because } \sum w_i = 0 \]

**Normalized Period Matrices:**

By the rank, \( A \) is nonsingular. We can find a unique basis of \( H^0(\mathbf{X}) \) s.t. \( A = \text{Id} \), \( \sum_{a_i} w_j = \delta_{ij} \).

Define \( Z = (\sum_{a_i} w_j) \), formally \( B \),

\[ \Lambda = H_1(\mathbf{X}, \mathbf{Z}) \in H^0(\mathbf{X})^* \Rightarrow (\Lambda, Z) \]

**Thm (Riemann Bilinear Relations):**

1. \( Z \) is symmetric, \( ^tZ = Z \)

2. \( \text{Im} \ Z > 0 \) (positive definite)
Remark: Consider any lattice \( \Lambda \in \mathbb{C}^g \). We can choose a basis s.t. \( \Lambda = (I_g, \mathbb{Z}) \). Then \( \Theta \) and \( \Theta' \) imply \( \mathbb{C}^g / \Lambda \) can be embedded in projective space.

Proof: (1) Choose a normalized basis \( w_1, \ldots, w_g \in H^1(\mathbb{Z}) \). Apply the basic period relation to \( \sigma = w_i, \tau = w_j \):

\[
\int_{\Delta} f_{w_i, w_j} = \sum_{k=1}^g \left( A_k(w_i) B_k(w_j) - A_k(w_j) B_k(w_i) \right)
\]

\[
= B_i(w_i) - B_j(w_j) = z_{ij} - z_{ji}
\]

But

\[
\int_{\Delta} f_{w_i, w_j} = \int_{\Delta} d(f_{w_i, w_j}) = \int_{\Delta} (df_{w_i} \wedge w_j + f_{w_i} \wedge dw_j)
\]

\[
= \int_{\Delta} (w_i \wedge w_j) + \text{to .}
\]

Then \( w_i \wedge w_j \) is a \((2,0)\) form, which they are now.

So \( \int_{\Delta} f_{w_i, w_j} = 0 \), i.e., \( z_{ij} = z_{ji} \).

(2) For any \( w_i \), \( \text{Im} \left( \sum_{k=1}^g A_k(w_i) B_k(w_i) \right) > 0 \).

Fix \( \lambda_1, \ldots, \lambda_g \in \mathbb{R} \), not all 0. Then we need to show

\[
\sum_{\alpha, \beta=1}^g \lambda_\alpha \lambda_\beta \text{Im} (B_{\beta}(w_\alpha)) > 0.
\]
Take \( w = \lambda_1 w_1 + \ldots + \lambda_j w_j \). Then

\[
\text{Im} \left( \sum_{k=1}^2 \overline{A_k(w)} B_k(\omega) \right) > 0
\]

\[
\Rightarrow \sum_{k,j} \lambda_k \lambda_j \text{Im} (B_k(\omega_j)) = \sum_{k,j} \lambda_k \lambda_j \text{Im} (B_k(\omega_j)) > 0.
\]

**Summary:**

- Choose symplectic basis \( \alpha, \beta \in H_1(\mathbb{X}, \mathbb{Z}) \).

- \( \exists! \) basis \( w_1, \ldots, w_j \in H^{1,0}(\mathbb{X}) \), \( \int w_j = \delta_{ij} \).

- \( \mathbb{Z} = \left( \sum_{b_i} w_j \right) \).

- RBI: \( t \mathbb{Z} = \mathbb{Z} \).

- \( \text{Im} \mathbb{Z} > 0 \).

**Example:** \( g = 1 \)

\( \mathbb{X} = \mathbb{C}/\Lambda \), \( \omega = dz \)

\[
\int_a^b dz = 1, \quad \text{take } z \text{ from } 0 \text{ to } 1
\]

\[
\int_a^b dz = \pi \quad \text{Need } z \text{ in upper half plane}
\]
Recall: Representing on $\mathbb{R}^3$ by $\mathbf{a}_i \cdot \mathbf{x}$.

Normalized basis 1-forms

$\omega_i, \ldots, \omega_j \in \Omega^1(\mathbb{R})$

$\int \omega_j = \delta_{ij}$

$\left( \int_{b_i} \omega_j \right) = (Z_{ij}) = Z$

Riemann bilinear relations:

1. $\bar{Z} = Z$
2. $\text{Im}(Z) > 0$.

$\int_{\Delta} f \bar{z} = \sum \left( \int_{a_k} \int_{b_k} - \int_{b_k} \int_{a_k} \right)$

$\overline{\omega}$ meromorphic, o.h.o.

Abel's Thm: Harder half.

$D$ a divisor of degree 0, $D = \Sigma n; P_i$ st. $\Sigma n; u(P_i) = 0 \in \text{Jac}(\mathbb{R})$.

Thm: $f$ meromorphic st. $\text{div}(f) = D$.

Main Claim: $f$ meromorphic, $\eta$ w/ simple poles (only) at $P_i$, st.

1. $f_{P_i}(\eta) = n_i$
2. $\int_{\eta} \in \mathbb{C} \cdot T \mathbb{R}$ for all $\gamma \in \Omega^1(\mathbb{R}, \mathbb{R})$ (Think $\eta = \frac{df}{\bar{z}}$).

Grant the existence of $\eta$. Define $f(x) = \int_{\gamma} \eta$, $x \in \mathbb{R} \setminus \{P_i\}$.

This is well defined by $\circ$, i.e., indep of choice of paths.
Locally near $P_i$, \( \eta = \frac{n_i}{z} + \text{holo} \). \( \text{Let} \)
\[
\int_{P_i}^z \eta = n_i \log z + \text{holo},
\]
\[
\int_{P_i}^z \eta = z^{n_i} \text{holo, mon. near at } P_i.
\]

So \( \text{ord}_{P_i}(\eta) = n_i \).

The issue is to construct \( \eta \).

**Step 2:** Given a divisor \( D = \sum n_i P_i \) of degree \( 0 \), find a meromorphic \( \eta \) with simple poles (only) at \( P_i \), \( \text{res}_{P_i}(\eta) = n_i \).

**Proof:** Consider
\[
\begin{align*}
\begin{cases}
\text{merom. diff} \\
\text{with simple poles at } P_i's
\end{cases}
\rightarrow & \mathbb{C}^d \\
& \text{for } (\lambda_1, \ldots, \lambda_d)
\end{align*}
\]

\[
\eta \rightarrow (\text{res}_{P_1}(\eta), \ldots, \text{res}_{P_d}(\eta)).
\]

By the residue theorem,
\[
\text{Im}(\alpha) \subseteq V = \left\{ (x_1, \ldots, x_d) \mid \sum x_i = 0 \right\}.
\]

**Claim:** \( \text{Im}(\alpha) = V \).

**Proof:** Count dimensions:
\[
\ker(\alpha) = H^0(\mathbb{C}) \text{ has dim } 1.
\]
\[
\text{dim LHS} = \mathbb{C}(K + P_1 + \ldots + P_d) \quad (2,1),
\]
\[
= (2g-2+d) + (1-g)
\]
\[
= g + d - 1
\]

So \( \text{dim Im} \alpha = d - 1 = \text{dim } V \). i.e., \( \alpha \) is onto.
Step 3: Assume mon \( \sum_{i=1}^{d} n_i \cdot u_i = 0 \) in \( \mathfrak{S}_n \).

Choose \( \eta = 0 \) in Step 2, \( \text{res}_t(\eta) = n_i \).

Main Point: Show that by adding holes differing to \( \eta \), we can arrange
\[ \int_{a_i} \eta, \int_{b_i} \eta \in \mathfrak{S}_n \cdot \mathbb{Z} \quad (1 \leq i \leq g) \]

- Fix minimalized basis \( u_1, \ldots, u_g \in H^{1,0}(E) \).
  \[ \int_{a_i} u_j = \delta_{ij}. \]

- And \( \exists \Delta = \sum_{k=1}^{g} u_k \), then \( \exists \Delta = \sum_{k=1}^{g} \eta_k \quad (1 \leq k \leq g) \).

Substep 1: Replacing \( \eta \) by \( \eta = \sum_{k=1}^{g} (\int_{a_i} \eta) u_k \), we can assume all \( \int_{a_k} \eta = 0 \), \( 1 \leq k \leq g \).

Substep 2: Basic period relation:
\[ \int_{a_1} \left( f_{u_k} \cdot \eta \right) = \sum_{k=1}^{g} \left( \int_{a_k} \left( \int_{a_k} \eta - \int_{b_k} \eta \right) \right) \quad \forall 1 \leq k \leq g. \]
\[ = \int_{b_k} \eta \quad \text{(because \( u_k \) is dual basis)} \]

But, on the other hand:
\[ \int_{a_1} f_{u_k} \cdot \eta = 2\pi i \sum_{k=1}^{d} f_{w_k} (P_k) \text{res}_t(\eta) \quad \text{(by residue Thm)} \]
\[ = 2\pi i \sum_{k=1}^{d} n_k \cdot f_{w_k} (P_k) \]

\[ \Rightarrow \]
\[ 2\pi i \sum_{k=1}^{d} n_k \int_{a_1} f_{w_k} = \int_{b_k} \eta \quad \forall 1 \leq k \leq g. \]
**4. Hypothesis:** \( \sum_{k=1}^{n} u_{ik}(p_k) = 0 \) means \( \exists e_1, \ldots, e_g \), \( f_1, \ldots, f_g \) s.t. \( \frac{d}{dx} \sum_{k=1}^{n} n_k \int_{p_k}^{p} = \sum_{i=1}^{g} \left( e_i \int_{a_i}^{b_i} + f_i \int_{b_i}^{p} \right) \) (**) 

As functions in \( H^{1,0}(B) \).

**Hit \( u_k \) with (**):** For each \( 1 \leq \alpha \leq g \)

\[
\frac{d}{dx} \sum_{k=1}^{n} n_k \int_{p_k}^{p} u_k = \sum_{i=1}^{g} \left( e_i \int_{a_i}^{b_i} + f_i \int_{b_i}^{p} \right)
\]

\[
= e_\alpha + \sum_{i=1}^{g} f_i \int_{b_i}^{p} \]

\[
= e_\alpha + \sum_{i=1}^{g} f_i \int_{b_\alpha}^{p} \text{ (Riem. B1.4).}
\]

**Upshot:** We had

\[
2\pi \sqrt{-1} \sum_{k=1}^{n} n_k \int_{p_k}^{p} u_k = \int_{b_k}^{p} \eta \quad \forall \alpha.
\]

So for each \( 1 \leq \alpha \leq g \),

\[
\int_{b_\alpha}^{p} \eta = 2\pi \sqrt{-1} \left( e_\alpha + \sum_{i=1}^{g} f_i \int_{b_i}^{p} \right)
\]

**4. Hypothesis:** Let \( \eta' = \eta - 2\pi \sqrt{-1} \left( \sum_{i=1}^{g} f_i \int_{b_i}^{p} \right) \)

\[
\int_{b_k}^{p} \eta' = 2\pi \sqrt{-1} e_\alpha \in 2\pi \sqrt{-1} \cdot \mathbb{Z}.
\]

\[
\int_{a_\alpha}^{b_\alpha} \eta' = -2\pi \sqrt{-1} \left( \sum_{i=1}^{g} f_i \int_{a_i}^{b_i} \right) = -2\pi \sqrt{-1} f_\alpha \in 2\pi \sqrt{-1} \cdot \mathbb{Z}.
\]
\[ \text{Div}^0(\mathcal{X}) \xrightarrow{u} \text{Jac}(\mathcal{X}) \].

We've shown \( \ker(u) = \text{Princ}(\mathcal{X}) \).

Now we just need to show \( u \) is surjective, the Jacobi inversion thm.
Structure of Abel-Jacobi Map:

\[ X = \text{curve of genus } g. \]

\[ \bar{u} : \text{Div}^g(X) \rightarrow \text{Jac}(X) \]

\[ \Sigma_{i} u(P_i) \rightarrow \Sigma_{i} u(P_i) \]

**Abe's Thm:** \[ \ker \bar{u} = \text{Princ}(X) \]

**Thm (Jacobi domain II):** \( \bar{u} \) is surjective. In particular, \( \text{Jac}(X) = \frac{\text{Div}^g(X)}{\text{Princ}(X)} \)

**Lemma:** Let \( P_1, \ldots, P_g \in X \) be \( g \) general points on \( X \), then \( P_1 + \cdots + P_g \) is not linearly equivalent to any other effective divisor, i.e.,

\[ \ell(P_1 + \cdots + P_g) = 1. \]

**Proof:** By R.R., \( \ell(P_1 + \cdots + P_g) > 1 \iff \ell(K - P_1 - \cdots - P_g) > 1 \).

Consider \( \phi_i : X \rightarrow \mathbb{P}^{g-1} \). Call \( \phi_i \Phi_i \).

\[ x \rightarrow [w_1(x), \ldots, w_{g-1}(x)] \]

\[ \ell(K - P_1 - \cdots - P_g) > 1 \iff \Phi(P_1), \ldots, \Phi(P_g) \text{ lie on a hyperplane in } \mathbb{P}^{g-1}. \]

Choose \( P_1, \ldots, P_g \in X \) whose images don't lie on a hyperplane in \( \mathbb{P}^{g-1} \). Then \( \ell(K - P_1 - \cdots - P_g) = 0 \). \( \blacksquare \)

**Exercise:** If \( D \) is a "general" effective divisor of \( \deg D \), then

\[ \dim \text{Id}_{D} = \begin{cases} 0 & \text{if } \deg \geq g \\ \deg & \text{if } \deg < g \end{cases} \]

Now define \( \nu : \text{Jac}(X) \rightarrow \text{Jac}(X) \) \( (g \text{ copies}) \)

\[ (P_1, \ldots, P_g) \rightarrow \sum_{i=1}^{g} u(P_i) \]
Thm (Jacobi dimension II): $v_3$ is surjective

"Jacobi $\Rightarrow$ Jacobi I": Need $\tilde{u}: \text{Div}^d(X) \to \text{Jac}(X)$ is onto.

Let $F \in \text{Jac}(X)$. Then Jacobi II $\Rightarrow F = \sum_{i=1}^2 u(P_i)$ for some $P_i$.

Then $F = \tilde{u}(\sum P_i - g P_0)$, since $u(P_0) = 0$.

Proof (Jacobi III): We use the fact (Remmert Proper mapping Thm) that the image $Z = v(X)$ of proper map $V$ is an analytic subvariety of $\text{Jac}(X)$ and Thm on dimension of fibers remains true.

Need to show $\text{Jac}(X) \cong Z = v(X)$, $Z$ has dim $g$.

If dim $Z < g$, then all fibers of $v$ would have dim $\geq 1$.

But $v(P_1, \ldots, P_g) = v(Q_1, \ldots, Q_g) \iff \sum u(P_i) = \sum u(Q_i)$

$\iff \tilde{u}(\sum P_i - \sum Q_i) = 0$

$\iff \sum P_i = \sum Q_i$ (Abel-Jacobi Thm)

However, the lemma tells us that if $P_i$ are general, then $\sum P_i$ is not linearly equivalent to any other effective divisor.

As general fibers of $v$ have dimension 0.

Fix $d > 0$, $P_0 \in X$.

Claim: We can identify $\text{Jac}(X) \cong \text{Div}^d(X) / \sim$, where $\sim$ is linear equivalence classes of degree $d$.

Use: $\text{Div}^g(X) \xrightarrow{\sim} \text{Div}^d(X)$ by $D \mapsto D + d P_0$. 
And Abel’s Thm that $\text{Jac}(\mathcal{X}) \cong \text{Div}^0(\mathcal{X})/\text{Prin}(\mathcal{X})$.

Now we want to construct a space parametrizing effective divisors of deg $d$.

**Symmetric Product:**

Fix $d > 0$, $\mathbb{X}^d = \mathbb{X} \times \ldots \times \mathbb{X}$ ($d$ times)

Symm grp $S_d$ acts by permuting the coordinates, so we can view $\mathbb{X}^d_{/S_d}$ as parametrizing unordered $d$-tuples, i.e.,

effective divisors of deg $d$.

**Thm:** $\text{cl} \mathscr{X}$ is a smooth curve, $\mathbb{X}^d_{/S_d} = \mathbb{X}_d = S^d(\mathbb{X})$ ($d^{th}$ symmetric product) is naturally a complex manifold of dim $d$ in such a way that

$$\mathbb{X}^d \longrightarrow \mathbb{X}_d$$

is a holomorphic map.

**Proof:** (Exercise) Hint: local coordinates on $\mathbb{X}_d$ are elementary symm forms in local coordinates on $\mathbb{X}$

Model: $\mathbb{C}^X = \mathbb{C}^d \longrightarrow \mathbb{C}_d = \mathbb{C}^d$

$$\begin{aligned}
(2, \ldots, 2) & \longmapsto (\text{elem symm forms})
\end{aligned}$$

**Exercise:** $S^d(\mathbb{P}^1) = \mathbb{P}^d$

**Note:** A point in $S^d(\mathbb{X})$ is an effective divisor of deg $d$ on $\mathbb{X}$, i.e.,

$$S^d(\mathbb{X}) = \left\{ \text{eff divs of deg } d \text{ on } \mathbb{X} \right\}$$
Remark: The Abel-Jacobi map gives a holomorphic map

\[ u_d : S^d(\mathbb{R}) \to \text{Jac}^d(\mathbb{R}) \]

\[ \begin{array}{c}
\mathbb{R}^d \\
\downarrow \\
S^d(\mathbb{R})
\end{array} \quad \begin{array}{c}
\text{div cl assoc at egd} \\
u_d
\end{array} \quad \begin{array}{c}
\text{Jac}(\mathbb{R})
\end{array} \]
**Thm (Abel-Jacobi restated):** Fix any integer \( d \). Then we may view \( \text{Jac}^d(\mathbb{P}^1) \) as parameterizing linear equivalence classes of divisors of degree \( d \) on \( \mathbb{P}^1 \). Write \( \text{Jac}^d(\mathbb{P}^1) \) (\( \equiv \text{Jac}(\mathbb{P}^1) \)).

Why? \( \text{Jac}(\mathbb{P}^1) = \begin{cases} \text{lin equiv} \ \{ \text{classes of deg } 0 \} \to \{ \text{lin equiv} \ \{ \text{of deg } d \} \} \\
\begin{array}{c}
D \mapsto D + dP_0
\end{array}
\end{cases} \)

Assume \( d \geq 1 \).

**Recall:**

\[
S^d(\mathbb{P}^1) = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \left/ \sum_{i=1}^{d} \text{symmetric group} \right.
\]

This is called the \( d \)th symmetric product of \( \mathbb{P}^1 \). The points of \( S^d(\mathbb{P}^1) \) are unordered \( d \)-tuples of points (repetitions allowed).

= effective divisors of deg \( d \) on \( \mathbb{P}^1 \).

**Thm:** \( S^d(\mathbb{P}^1) \) is a complex manifold of complex dimension \( d \).

Moreover, \( u_d : S^d(\mathbb{P}^1) \to \text{Jac}^d(\mathbb{P}^1) \) is a holomorphic map.

\[
\begin{array}{c}
D \mapsto [D]
\end{array}
\]

\[
u_d : S^d(\mathbb{P}^1) \to \text{Jac}^d(\mathbb{P}^1)
\]

\[
d \mapsto d
\]

Fix a linear equivalence class \([D] \in \text{Jac}^d(\mathbb{P}^1)\). (deg \( D = d \)).

\[
u_d^{-1}( [D] ) = \begin{cases} \text{effective divisors } D' \mid D' = D \end{cases}
\]

\[
= |D| \text{ complete linear series associated to } D.
\]
Thm: \( U_d^{-1}(P^1,\mathbb{P}^r) \) is a projective space as a subvariety of \( S^d(X) \).

Structure of \( U_d \):

1. \( d \leq g \): We saw that the general fiber of \( U_d \) is finite, hence the general fiber is a point. Here \( U_d \) is birational onto its image.

2. \( d > g \): \( U_d \) is surjective. The general fiber has \( \dim d - g \).
   
   i.e., if \( |D| \) is a general divisor class of \( \deg d > g \), then \( \dim |D| = d - g \). Riemann-Roch says \( \ell(D) = \dim |D| + 1 \)  
   
   \[ \ell(D) = d + 1 - g + \ell(K-D) \]
   
   i.e., \( \dim |D| = d - g + \ell(K-D) \). For \( d \geq g \), one can show \( \ell(K-D) = 0 \) for general \( D \).

3. \( g < d \leq 2g-2 \): Most fibers of \( U_d \) are \( \mathbb{P}^{d-g} \)'s, but some have \( \dim > d - g \), i.e., the ones where \( \ell(K-D) > 0 \).

4. \( d > 2g-2 \): \( \ell(K-D) = 0 \) for all \( D \). Then all fibers are \( \mathbb{P}^{d-g} \).

In fact: \( S^d(X) \to \text{Jac}(X) \) is a proj space bundle (in Zariski top).

Let's go back to \( d \leq g - 1 \):

Defi: \( W_d = \text{im} (U_d) \)  
\[ W_d = \mathcal{I} \text{ linear equis classes of } \mathcal{O}_X \text{ divs of } \deg d \]
\textbf{Example: } \quad d = 0, \quad (g \geq 2)

\[ S^0(X) \rightarrow \text{Jac}^0(X) \]

- $X$ is mon-hyperelliptic, then $U_0$ is an embedding
- $X$ hyperelliptic, $\exists \ X \xrightarrow{\mathcal{P}^1}{\mathbb{P}^1}$ (unique)
  
  Then $p^{-1}(pt) = \text{climian class no. degree } \geq 2 \vee r(D) = 1.$
  
  i.e., $p^{-1}(pt) = \mathbb{P}^1.$

\[ \xymatrix{ \mathbb{P}^1 \ar@{.>}[d] & S^0(X) \ar[l] \ar[r] \ar[d] & \text{Jac}^0(X) \ar[l] \ar[d] \ar[r] & W_0 \subseteq \text{Jac}^0(X) \ar[l] } \]

\[ g = 2: \] Every curve is hyperelliptic

\[ \xymatrix{ \mathbb{P}^1 \ar@{.>}[d] & S^0(X) \ar[l] \ar[r] \ar[d] & \text{Jac}(X) \ar[l] \ar[d] } \]

\textbf{Example: } \quad d = g - 1.

\[ S^{g-1}(X) \]

\[ \xymatrix{ \ast \ar@{.>}[d] & S^{g-1}(X) \ar[l] \ar[r] \ar[d] & \text{Jac}^{g-1}(X) \ar[l] \ar[d] } \]
Next Big Topic: Find equation for $W_{g-1}$ (Riemann theta-fcn)

$g=4$: $d = g-1 = 3$

$S^3(\mathbb{R})$

$W_3 \subseteq \text{Jac}^3(\mathbb{R})$

$\dim 4$

$\mathbb{P}^1 \rightarrow \mathbb{P}^1$

Rulings of quadric in complex space

$^*$

$W_3$ for general $\mathbb{R}$ has 0 sing points.

$f_2(x) + f_3(x) + \ldots$

"Mumford's Curves and Their Jacobians"
Recall:

\[ X = \mathbb{P}^1 \text{ genus } g \geq 2 \]

\[ \text{Div}_d : \mathbb{P}^1(K) \to \text{Jac}^d(X) \]

\[ \text{Div}_d((D)) = \text{ID} = \mathbb{P}^{\text{dim}(g)} \]

For \( d \geq g-2 \):

\[ W_d = \text{Div}_d \left( \mathbb{P}^1(K) \right) \subset \text{Jac}^d(X) \]

\( W_d \) parameterizes all effective divisor classes \( 0 \leq d \).

Example: Any \( RS \) of genus \( g \geq 2 \) can be expressed as a branch covering \( \pi : \mathbb{X} \to \mathbb{P}^1 \) of \( \text{deg } g \).  

\[ \text{PF:} \]

Need \( D \) of \( \text{deg } g \) s.t. \( \text{l}(D) \geq 2 \), \( \text{r}(D) = \text{l}(D) - 1 \geq 1 \).

If \( \text{deg } D = g \), then \( \text{l}(D) = g + 1 - g = l(K-D) \) (PE)

\[ = 1 + l(K-D) \]

We need \( l(K-D) \geq 1 \).

\[ \text{deg } (K-D) = (g-2) - g = g-2 \]

Need \( K-D = E \) effective divisor of degree \( g-2 \).

Take \( E \in W_{g-2} \), \( D = K - E \).

In fact, have \( g-2 \)-dimensional family of linear equivalents classes of \( D \)’s with \( l(D) \geq 2 \).

Example (Riemann’s count): "Compute" \( \dim \left\{ X | \text{X has genus } g \geq 2 \right\} \).

Take \( d \gg 0 \). Consider

\[ Z = \left\{ (X, \pi) \mid X \text{ is genus } g, \pi : X \to \mathbb{P}^1, \text{deg } d \right\} \]

Take \( (X, \pi) \in Z \). \( \pi : X \to \mathbb{P}^1 \).

Suppose \( \pi \) has only simple branching, i.e., \( c_\pi(x) = 1,2 \) for all \( x \in X \). All ramification points have distinct images on \( \mathbb{P}^1 \).
# branch points \( \beta \) \((\partial g-2)+2d=b \)

As we get \( b \) points in \( \mathbb{P}^1 \), i.e., an effective divisor, \( \mathcal{d} \in \mathbb{P}^b \).

\[
\mathcal{Z} \to \mathbb{P}^b
\]

\((X, \pi) \to \text{branch divisor of } \pi = \text{Br}(\pi)\).

**Claim:** \( \beta \) is dominating and generically finite

**Idea:**

Branch cover gives \( \tau_1, \ldots, \tau_b \in \Sigma_d \sim \text{symm grp} \)

transpositions i.e. \( \tau_1, \ldots, \tau_b = \{1\} \). Conversely, given such permutations \( \tau_1, \ldots, \tau_b \) one can build \( X \to \pi \to \mathbb{P}^1 \).

As for any \( b \) distinct points, can construct finitely many \( X \)s.

\[
\mathcal{Z} = \tilde{\mathcal{Z}} \text{ or } (X, \pi) \text{ genus } g, \pi : \tilde{X} \to \mathbb{P}^1 \text{ deg } d \text{ up to change of coords on } \mathbb{P}^1
\]

As \( \dim \mathcal{Z} = b - 3 \leftarrow \text{dim} \text{ PG}_L(1) \)

\[= (\partial g - 5) + 2d \]
We also have
\[
\mathcal{Z} \xrightarrow{(X, \pi)} (X, \pi) \xrightarrow{\pi} M_g
\]
\[\{X, \pi\} = M_g \xrightarrow{\pi} \mathcal{Z}\]

We need to compute the dimension of fibers of \(Z \rightarrow M_g\),
i.e., for given \(X\), what is the dimension of set of all \(\pi: \mathcal{Z} \rightarrow \mathbb{P}^1\) for \(d \gg 0\)

\[\mathcal{Z} = \left\{ \text{linear series on } \mathcal{Z} \right\}_{\text{of degree } d \text{ and dimension } 2}\]

\[S^d(X) \rightarrow \mathbb{P}^d\text{-bundle}\]
\[\text{Fix } [D] \in \text{Jac}^d(X), \quad [D] = \mathbb{P}^d\]

\[\dim \text{Grass}(\mathbb{P}^1, \mathbb{P}^d) = 2(d - g - 1)\]

\[\dim \left\{ 1\text{-dim linear series on } \mathcal{Z} \text{ of degree } d \gg 0 \right\} = 2(d - g - 1) + g\]

For fixed \(X\), \(\dim \left\{ \pi: \mathcal{Z} \rightarrow \mathbb{P}^1 \right\} = 2d - g - 2\)

\[\mathcal{Z} = \left\{ (X, \pi) \right\} \leq \dim \Theta_{g-5+2d}\]
\[\text{and } \dim = 2d - g - 2\]
A0, \[ \dim W_g = (2g-5+2d) - (2d-g-2) \]
\[ = 3g-3. \]

\[ W_d \in \text{Jac}^d(X) \]

\[ W_d^r(X) = \left\{ \mathcal{D} \mid r(\mathcal{D}) \geq r \right\} . \]

\[ S^d(X) \quad \text{Can think of} \]
\[ \downarrow u_d \]
\[ \text{Jac}^d(X) \]

\[ W_d^r(X) = \left\{ \mathcal{D} \mid \dim u_d^r(\mathcal{D}) \geq r \right\} . \]

**Thm ("Beauville-Nakai"):**

1. For any \( X \), \( \dim W_d^r(X) \geq g - (r+1)(g-d+r) \)

2. For "general" \( X \), \( \dim W_d^r = g - (r+1)(g-d+r) \)
   (expt: \( r \) s.t. \( \text{RHS} < 0 \), then \( W_d^r = \emptyset \)).

**Example:** For \( X \) of genus \( g \), what is least \( d \) for which \( \exists X \rightarrow \mathbb{P}^r \deg d \)?

**Solution:** Need \( d \) s.t. \( W_d^r(X) \neq \emptyset . \)

\( r = 1: \)
\[ g - 3(g - d + 1) \geq 0 \]
\[ 3d - 2 \geq g \]
\[ d \geq \frac{g + 2}{3} \]

\( g = 2: \)
\( d \geq 2 \)
\( g = 3: \) \( d \geq 3 \) \( \dim W_3^2(X) \geq 1 \)
Recall: if \( g > 1.5 \) genus \( g \), we can write period matrix \( \tau \) \((\text{Ig}, \bar{z})\),
\[ \tau \in \mathbb{C}, \quad \text{Im}(\tau) > 0. \]

Fix an arbitrary \( g \times g \) complex matrix \( \Omega \) satisfying
\[ \Omega^{T} \Omega = \Omega, \]
\[ \text{Im} \Omega > 0. \]

We will define and study Riemann \( \Theta \)-function
\[ \Theta(z, \Omega) = \Theta(z) \]
an entire function on \( \mathbb{C}^g \).

**Def:** Let \( \mathbb{C}^g \otimes \Lambda_\Omega = \left\{ \text{Lattice generated by columns} \right\} \)

We'll see that \( \mathbb{C}^g / \Lambda_\Omega \) is \( \Lambda_\Omega \)-invariant, so defines a hypersurface
\[ \Theta \text{ in } A_\Omega = \mathbb{C}^g / \Lambda_\Omega. \]

When \( \Omega \) is normalized period matrix of \( g \times g \), we will prove
\[ \Theta = W_{g}, \quad (\text{up to translation}) \]
\[ \subseteq A_\Omega = \text{Jac}(\mathcal{X}). \]

We'll prove Torelli's Thm: curve \( \mathcal{X} \) is uniquely determined by \( (\text{Jac} \mathcal{X}, \Theta) \).

Fix \( \Omega \) as above: \( \tau \Omega = \omega, \quad \text{Im} \omega > 0. \)
Prop/Def: The infinite series $\mathcal{F}(z, \Omega) = \sum_{n \in \mathbb{Z}^d} e^{\left(\pi i \lambda_n z \cdot \delta z + \pi i \lambda_n \cdot \delta z\right)}$

converges absolutely and uniformly on compact subsets of $\mathbb{C}^d$ to define an entire function $\mathcal{F}(z, \Omega)$.

Proof: Since $\Omega$ satisfies the Kiemann conditions, $\text{Im}(\lambda n \Omega) \geq c_2 \sum_{n \neq 0} n^2$ for some constant $c_2$. (Take $c_2$ s.t. $\text{Im} \Omega > 0, c_2$)

Any $\max \text{Im} z \leq C_0$. Then

$$|e^{\left(\pi i \lambda_n z \cdot \delta z + \pi i \lambda_n \cdot \delta z\right)}| \leq e^{-c_2 \sum_{n \neq 0} n^2 + c_2 \Sigma 1_{n = 1}}.$$ 

So then we have $\mathcal{F}(z, \Omega) \leq C_3 \left(\sum_{n \neq 0} e^{-c_2 \sum_{n \neq 0} n^2}\right)^{1/3} < \infty.$

Prop: $\mathcal{F}(z+n) - \mathcal{F}(z)$

$\mathcal{F}(z+\Omega n) = e^{\left(-\pi i \lambda_n z \cdot \delta z - \pi i \lambda_n \cdot \delta z\right)} \cdot \mathcal{F}(z)$.

Note: $\mathcal{F} \neq 0$:

$\mathcal{F}(z)$ is periodic w.r.t. $\mathbb{Z}^d \subset \mathbb{C}^d$.

$\mathcal{F}(z) = \sum_{n \in \mathbb{Z}^d} e^{\pi i (\lambda_n z \cdot \delta z + \pi i \lambda_n \cdot \delta z)}$ is its Fourier exp.

Coefficients are $\neq 0 \Rightarrow \mathcal{F} \neq 0$.

Corol: Zero locus $\mathcal{F} = 0$ is invariant under translation by $\Lambda_n$. So we get a naturally defined divisor

$D = \Lambda_n : \mathcal{F}^\gamma/\Lambda_n$. 
Remark: Can use $\nu$-functions (and cousins) to define projective embeddings $A_\omega \hookrightarrow \mathbb{P}^n$.

Summary: $\Omega \in \mathcal{T} \cap \mathcal{J}$, $\text{Im}(\Omega) > 0$ gives us:

- $\Lambda_\Omega \in \mathbb{C}^g$
- $A_\Omega = \mathbb{C}^g / \Lambda_\Omega$ complex torus
- $\Theta_\Omega \in A_\Omega$ divisor, actually an ample divisor of minimal class.
- $A_\Omega$ a proj alg. variety.

$(A, \Theta)$ is called a principally polarized abelian variety.

Aside: Prove mult$_\mathbb{Q}$(\Theta) $\infty$ $\forall x \in A_\Omega$ (Thm by Kollár)

Back to R.S.:

$\Delta = $ compact R.S. genus $g$.

$\omega_1, \ldots, \omega_g \in H^0(\Delta)$ normalized basis.

$\Delta \xhookrightarrow{\psi} \mathbb{C}^g$

$\psi \downarrow \Delta \xrightarrow{\psi} \mathbb{C}^g$

Have "Abel - Jacobi" map $\tilde{\psi}_*: \Delta \rightarrow \mathbb{C}^g$ $\psi_*: \psi_* = (\omega_1, \ldots, \omega_g)$.

Not uniquely defined up to translation. To compensate for the choice of $p_o$, we'll look at translates.

Fix $x \in \mathbb{C}^g$. 
Define analytic eta on \( \Delta \)

\[
\tilde{\eta}_x(p) = \eta\left(x + \int_{p_0}^p \omega \right) \\
= \eta\left(x + \tilde{u}(p) \right)
\]

Riemann's Thm: \( \exists \delta \in C^g \) s.t for fixed \( x \in C^g, \tilde{f}_x(p) \) either

vanishes identically on \( \Delta \) or else has \( g \) zeroes

\( Q_1, \ldots, Q_g \in \Delta \) (counting multiplicity) s.t.

\[
\sum_{i=1}^g \int_{p_0}^{Q_i} \omega = -x + \delta \pmod{\Lambda_G}
\]

i.e., \( \tilde{u}(Q_1, \ldots, Q_g) = [-x + \delta] \in \text{Jac}(\mathbb{B}) \).
\[ X = \mathcal{R}, \text{ genus } g, \omega = (\omega_1, \ldots, \omega_g) \text{ normalized periods} \]

\[ \Delta = \text{period matrix, } \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_g \end{pmatrix} \]

\[ \mathcal{V}(z) = \sum_{n \in \mathbb{Z}^g} e^{(2\pi i n \cdot \omega + 2\pi i n \cdot \omega)} \text{ an entire function in } \mathbb{C}^g \]

\[ \Lambda = \Lambda_\Delta = \{ \mathbf{v} \text{ lattice spanned by } \omega \} \]

\[ \text{Jac}(\Lambda) = \text{Jac}(\Delta) = \mathbb{C}^g / \Lambda \]

\[ \mathcal{V}(z+1) = \mathcal{U}_g(z) \mathcal{V}(z) \text{ all } z + \Lambda, \mathcal{U}_g \text{ matrix vanishing} \]

\[ \Theta := \int_{\Delta} \mathcal{V} = 0 \leq \text{Jac}(\Delta). \]

**Remark:** \[ \mathcal{V}(z) \] is an even function.

\[ \Delta \]

Given \( x \in \mathbb{C}^g \), consider the function on \( \Delta \):

\[ \int_{\Delta} \mathcal{V}(x) = \mathcal{V}(x + \sum_{n} \omega) = \mathcal{V}(x + u(x)) \]

**Riemann's Thm:** There exists \( \delta \in \mathbb{C}^g \) s.t. for any fixed \( x \in \mathbb{C}^g \), we have \( \int_{\Delta} \mathcal{V}(x + u(x)) \) either vanishes identically or has \( g \) zeros counting multiplicity of \( Q_1, \ldots, Q_g \) s.t.

\[ \sum_{i=1}^{g} \int_{\Delta} \mathcal{V}(x + u(x)) = -x + \delta \pmod{\Lambda} \]

**We'll prove this theorem next time. Today we deduce geometric consequences.**
Fix any $a \in J(\mathcal{X})$. Let $\mathcal{X}_a = \mathcal{X} + a$ translate of $\mathcal{X}$ by $a$.

Note: The images of $f_a$ are well-defined on $\mathcal{X}$. The defo imply

$$
(\begin{array}{c}
\text{zeros of } f_a \\
\text{zeros of } a
\end{array}) = (\begin{array}{c}
\mathcal{X}_a \\
\emptyset
\end{array})
$$

For most $a$, $\mathcal{X}_a \cap \emptyset$ is a finite set.

As divisors,

$$\emptyset \cdot \mathcal{X}_a = Q_1 + \cdots + Q_g \leq \mathcal{X}(\mathcal{X})$$

Main point: $u(Q_1) + \cdots + u(Q_g) = -a + S$ in $J(\mathcal{X})$.

This is relating the group law in the Jacobian and the intersection of the theta divisor with $\mathcal{X}_a$.

Jacobi inverse: Since $S^g(\mathcal{X}) \longrightarrow \text{Jac}^g(\mathcal{X}) = \text{Jac}(\mathcal{X})$ is surjective and generically 1-1.

Given general $\beta \in J(\mathcal{X})$, $J(\mathcal{D}) \cap \psi \subseteq \mathcal{X}$, $\psi(\mathcal{D}) = \beta$. In fact,

$$\mathcal{D} = \emptyset \cdot \mathcal{X} + \psi$$
As we have an effective form of Jacobian inversion:

**Carl a. Riemann's Thm:** For any \( g \)-1 points \( p_1, \ldots, p_g \in \mathbb{X} \),

\[
u(p_1 + \cdots + p_g) - \delta \in \Theta, \text{ i.e., } \nu(\sum_{i=1}^{g} p_i \omega - \delta) = 0.
\]

**Remark:** \( U_{g-1} : S^{g-1}(\mathbb{X}) \longrightarrow J(\mathbb{X}) = S^g(\mathbb{X}) \)

\( W_{g-1} = \text{im}(U_{g-1}) \)

As the corollary, \( W_{g-1} \subseteq \Theta + \delta \).

**Pf (cont.):** Let it is enough to prove this for general points \( p_1, \ldots, p_g \in \mathbb{X} \)

\[
\begin{aligned}
(U_{g-1}^{-1}(\Theta + \delta) \subseteq S^{g-1}(\mathbb{X}) \text{ analytic subvariety, if it contains a general point of } S^{g-1}(\mathbb{X}) \),
\end{aligned}
\]

Choose \( g \) general points \( p_1, \ldots, p_g \in \mathbb{X} \) s.t. \( \lambda(p_1 + \cdots + p_g) = 1 \)

Consider \( \nu(\delta - \sum_{i=1}^{g} \int_{p_i} \omega + \sum_{i=1}^{g} \int_{p_i} \omega) = \psi(p) \) where

\[
y = \delta - \sum_{i=1}^{g} \int_{p_i} \omega.
\]

Suppose \( \psi \equiv 0 \). Then take \( p = p_0 \),

\[
\nu(\delta - \sum_{i=1}^{g} \int_{p_i} \omega) = 0.
\]

But \( \nu \) is even, so

\[
\nu(\sum_{i=1}^{g} \int_{p_i} \omega - \delta) = 0.
\]

Suppose \( \psi \neq 0 \). Then \( \psi \) has \( g \) zeroes \( Q_1, \ldots, Q_g \) s.t.

\[
\sum_{i=1}^{g} \int_{Q_i} \omega = \left( \sum_{i=1}^{g} \int_{p_i} \omega - \delta \right) + \delta. \text{ (Riemann's thm)}
\]

As \( U(Q_1 + \cdots + Q_g) = U(p_1 + \cdots + p_g) \), then using Abel’s thm we have \( Q_1 + \cdots + Q_g = p_1 + \cdots + p_g \).

Since \( \lambda(p_1 + \cdots + p_g) = 1 \) this means \( p_1 + \cdots + p_g = Q_1 + \cdots + Q_g \).

In particular, \( \psi(p_g) = 0 \Rightarrow \nu(-\sum_{i=1}^{g} \int_{p_i} \omega + \delta) = 0 \).
Recap:

\[ W_{g-1} \subseteq \Theta + \delta \]

Next time:

- \[ W_{g-1} = \Theta + \delta \]
- Prove Riemann's thm.
\[ \overline{\Delta} \]

\[ \mathfrak{X} = \mathbb{R}^3 \text{ genus } g \]

\[ w_1, \ldots, w_g \text{ normalized basis by diff.} \]

For \( x \in \mathbb{C}^g \), define \( f_x(p) = \mathcal{V}(x + \sum_{p_0} w) \)

**Thm:** \( \exists \mathcal{S} \subset \mathbb{C}^g \) s.t for all \( x \in \mathbb{C}^g \), either \( f_x \) vanishes identically or else has \( g \) zeroes \( q_1, \ldots, q_g \) s.t.

\[ \sum_{q_i} \frac{1}{\mathfrak{S}_0} \sum_{p_0} \overline{w} \equiv -x + \mathcal{S} \pmod{\mathfrak{A}}. \]

**Proof:** We assume \( f_x \neq 0 \). Let \( \mathcal{S} = \{ q_i \} \) be the zeroes of \( f_x \) (in interior or \( \overline{\Delta} \)).

**Step I:** Show there are \( g \) zeroes counting multiplicities.

**Pf:**

\[ \# \text{ zeroes of } f \text{ in } \overline{\Delta} = \frac{1}{2\pi i} \int_{\partial \overline{\Delta}} \frac{df}{f} \]

\[ = \frac{1}{2\pi i} \left( \sum_{k \in A} \int_{A_k^+ - A_k^-} \frac{df}{f} + \sum_{k \in B} \int_{B_k^+ - B_k^-} \frac{df}{f} \right) \]

\[ \int \mathbb{C} - \mathbb{B}^2 \]

Want to compare \( \int \frac{df}{f} \) at \( p, p' \). (k is fixed here)

\[ \int_{p} w - \int_{p'} w = \int_{p} w = e_k \]

\[ f_x(p') = \mathcal{V}\left(x + \int_{p} w\right) \]

\[ = \mathcal{V}\left(x + \int_{p} w - e_k\right) \]

\[ = \mathcal{V}\left(x + \int_{p} w\right) \]

\[ = f_x(p). \]
\[ \int \frac{df}{f} = 0 \quad \text{since } f \text{ is the same in } B^+_k \text{ as } B_k. \]

\[ B^+_k \cap B_k \]

\[ \int_{A_k^*} \frac{df}{f} : \quad \text{Place } p \text{ and } p' \text{ similarly to in } B_k \text{ calculation.} \]

\[ \int_{p'}^p \omega - \int_{p}^{p'} \omega = \int_{p}^{p'} \omega \]

\[ = - \int_{B_k} \omega \]

\[ = -\Omega \omega \quad \text{where } \Omega \text{ is the period matrix.} \]

\[ \int_{x} (p') = \mathcal{V}(x + \int_{p}^{p'} \omega) \]

\[ = \mathcal{V}(x + \int_{p}^{p'} \omega + \Omega \omega) \]

\[ = \mathcal{V}(x + \int_{p}^{p'} \omega) \exp(-\pi i \Omega \omega - 2\pi i (x + \int_{p}^{p'} \omega) \cdot e_k) \]

\[ = \int_{x} (p) \exp(-\pi i \Omega \omega - 2\pi i (x + \int_{p}^{p'} \omega) \cdot e_k) \]

\[ d \log f(p) = d \log (f(p)) - 2\pi i \omega \]

\[ \int \frac{df}{f} = \int -2\pi i \omega = 2\pi i \int \omega \]

\[ A_k^* - A_k^* \quad A_k^* \quad A_k \]

\[ = 2\pi i \]

\[ A_0 \quad \frac{1}{2\pi i} \int \frac{df}{f} = g. \]

\( \square \)
Step 2: Fix index $l \leq j$. Let $g_e = \int_{p_l}^p w_e$, $dw_e = dg_e$.

Hence, on $\Delta$, $\frac{1}{2\pi i} \int_{\Delta} g_e \frac{df}{f} = \sum_{i} \text{res}_{\Delta_i} (g_e \frac{df}{f})$

$= \sum_{i=1}^{g} g_e \lambda_i \text{ord}_{\Delta_i} (\lambda_i)$

$= \sum_{i=1}^{g} \text{ord}_{\Delta_i} (\lambda_i) \int_{p_l}^{p_i} w_e$

Enough to show

$\frac{1}{2\pi i} \int_{\Delta} g_e \frac{df}{f} \equiv -\chi_\lambda + \delta_\lambda (\text{mod } \Lambda)$.

As before:

$\int_{\Delta} g_e \frac{df}{f} = \sum_{k=1}^{g} \left( \int_{B_k^+}^{B_k^-} + \int_{A_k^-}^{A_k^+} \right)$

$\int_{F_n} g_e \frac{df}{f}$: For $p \in B_k^-$, $p' \in B_{k'}^+$,

$g_e(p) - g_e(p') = \int_{p'}^{p} w_e$

$= \int_{A_k^-}^{A_k^+} w_e$

$= -\delta_{kk}$

We saw in Step 1 that

$f(p') = f(p)$

$\sum_{k=1}^{g} \int_{B_k^+-B_k^-} g_e \frac{df}{f} = \sum_{k=1}^{g} \delta_k f_k \int_{B_k^-}^{B_k^+} \frac{df}{f}$
\[
\int_{B_e^-} d \log f = \text{change in value of } \log f \text{ as you move along } B_e^-.
\]

\[
\begin{align*}
\frac{f(y')}{f(y)} &= \mathcal{F}(x + \int_{B_e^-} y' \omega) \\
&= \mathcal{F}(x + \int_{B_e^-} y' \omega + \int_{B_e^-} \omega) \\
&= \mathcal{F}(x + \int_{B_e^-} y' \omega - \Omega \epsilon_e) \\
&= \frac{f(y)}{f(y')} \exp\left( \pi i \Omega \epsilon_e + \partial_x \left( x_e + \int_{B_e^-} \omega \right) \right).
\end{align*}
\]

So the change in the value of \( \log f \) as you move along \( B_e^- = -\pi i \Omega \epsilon_e - \partial_x \left( x_e + \int_{B_e^-} \omega \right) = -\pi i x_e + \partial x_i m_k \) (for some \( m_k \)) is:

\[
-\partial x_i x_e + \text{stuff independent of } x_e + \partial x_i m_k.
\]

Make a similar computation for \( \int_{A_k^-} \).

So we get:

\[
\sum_{A_k^-} \partial x_i \int_{A_k^-} g \frac{df}{f} = (\text{term independent of } x) + \Omega \epsilon_e
\]

and

\[
\int_{A_k^-} \frac{1}{2 \pi i} \partial \int_{A_k^-} g \frac{df}{f} = -\gamma_k + \left( \text{stuff independent of } x \right) \pmod{\Lambda}.
\]
Let $K \subset \overline{K}$ be not necessarily algebraically closed.

Let $C/K$ be a curve. Suppose $D$ is a divisor over $K$. Suppose $\sigma \in \text{Gal}(\overline{K}/K)$. Then $D$ is defined over $K$ if $D^\sigma = D$ where $D = \sum_p \mathcal{O}_p$, then $D^\sigma = \sum_p \mathcal{O}_p \sigma(p)$.

For a rational function, $\text{div}(f^\sigma) = \text{div}(f)^\sigma$. So if $f \in K(C)$, then $\text{div}(f)$ is a divisor defined over $K$.

**Lemma:** Let $V$ be a $K$-vs., and assume $\text{Gal}(\overline{K}/K)$ acts continuously on $V$ in a manner compatible with its action on $K$. Let $V_K = \ker(\text{Gal}(\overline{K}/K)) = \{ v \in V | \forall \sigma \in \text{Gal}(\overline{K}/K), \sigma(v) = v \}$

Then $V \cong K \otimes_K V_K$; i.e., $V$ has a basis of $\text{Gal}(\overline{K}/K)$ invariant vectors.

**Prop:** Let $C/K$ be a smooth curve, and let $D \in \text{Div}_K(C)$. Then $L(D)$ has a basis consisting of functions in $K(C)$.

**Proof:** Since $D$ is defined over $K$, we have $L(D^\sigma) = L(D)$. So for $f \in L(D)$, $f^\sigma \in L(D) \forall \sigma \in \text{Gal}(\overline{K}/K)$. Thus $\text{Gal}(\overline{K}/K)$ acts on $L(D)$. Now apply the above Lemma to $L(D)$ to conclude it has a basis of functions in $K(C)$. 

Given a curve of genus 0, let \( p \in C \) be any point.

Then \( \deg K = 2g - 2 = -2 \) for any canonical divisor \( K \).

So, in particular, \( \ell(K - p) = 0 \) whenever \( \deg p > \deg K = -2 \).

Thus \( \ell(p) = \deg p + 1 - g + \ell(K - D) = \deg p + 1 = 2 \). As \( f \) is a nonconstant function in \( \mathbb{K}(p) \), by definition of \( \mathbb{K}(p) \),

\( f \) must have a simple pole at \( p \) and no others. We always have a map from any curve \( C \to \mathbb{P}^1 \) defined by \( P \mapsto [f(P):1] \). The degree of this map is the \# of preimages of any point \( p \). Since \( f \) has a single simple pole, we get this map having degree 1, hence is an isomorphism.

This same process would give \( C \cong \mathbb{P}^1(K) \) over an algebraically closed field using the results on curves over arbitrary fields.
Let $E$ be a curve of genus 1 and suppose $p$ a $k$-rational point $p \in E$. Then we can consider the vector spaces $V(p)$ for $n \in \mathbb{N}$. Let $div(u) = K - D$ be a canonical divisor, so it has deg $2g-2 = 2 \cdot 1 - 2 = 0$. As for any divisor $D$ we have deg $D > 0$, $l(K-D) = 0$.

$l(np) = n + 1 - 1 + l(K-np)$.

As for $n = 0$, we have $l(0,p) = 1$, thus $l(0,p)$ consists of constants.

<table>
<thead>
<tr>
<th>$np$</th>
<th>$l(np)$</th>
<th>Basis of $V(np)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.p</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.p</td>
<td>1</td>
<td>1, $x$</td>
</tr>
<tr>
<td>2.p</td>
<td>2</td>
<td>1, $x, y$</td>
</tr>
<tr>
<td>3.p</td>
<td>3</td>
<td>1, $x, y$</td>
</tr>
<tr>
<td>4.p</td>
<td>4</td>
<td>1, $x, y, x^2$</td>
</tr>
<tr>
<td>5.p</td>
<td>5</td>
<td>1, $x, y, x^2, xy$</td>
</tr>
<tr>
<td>6.p</td>
<td>6</td>
<td>1, $x, y, x^2, xy, x^3, y^2$</td>
</tr>
</tbody>
</table>

As there are 7 elements in $V(6,p)$, so there must be a relation amongst them. This will give the Weil-Petersson equation.

Call the equation this defines in $P^2 \mathbb{C}$, then we have a map $\phi: E \to C$ with $\phi = [x:y:1]$. We want $\phi$ to be an iso, so we need to show $K(E) = K(x,y)$. Since $x$ has degree 2, $[K(E):K(x,y)] = 2$, similarly $[K(E):K(y)] = 3$. Since 2 and 3 must be divisible by $[K(E):K(x,y)]$, it must be 1.