Essentially, AG is the study of geometric objects that locally look like zero-sets of polynomials.

The Elegant Universe, by Brian Greene

I. AFFINE ALGEBRAIC GEOMETRY

Fix a field $k$ (for concreteness $k = \mathbb{C}$).

Definition: Affine $N$-space over $k$, denoted $\mathbb{A}^N_k$ (or just $\mathbb{A}^N$) is the set of $N$-tuples with entries in $k$.

$\mathbb{A}^N_k = \{(a_1, a_2, \ldots, a_N) \mid a_i \in k\}$ (This is just $k^N$, but without any implied special point [like the origin] or vector space structure).

Definition: An affine algebraic variety $V$ is the common zero of an arbitrary collection of polynomials $\{F_\lambda\}_{\lambda \in \Lambda}$ where $F_\lambda \in k[x_1, x_2, \ldots, x_N]$.

Notation: $V = V(\{F_\lambda\}_{\lambda \in \Lambda}) \subseteq \mathbb{A}^N_k$

\[ \{\rho \in \mathbb{A}^N_k \mid \forall \lambda \in \Lambda, F_\lambda(\rho) = 0\} \]

Examples:

1. $V(y - x^2)$

Note: $k$ is a field, but for the purpose of doing proofs we allow $k = \mathbb{R}$.
\[ W(y^2 - x^2 - xy) \]
\[ W(z^2 - x^2 - y^2) \]

\[ W(xy + xz) \leq A^3 \]
\[ x=0 \text{ or } y = z \]

Recall, a field \( k \) is algebraically closed if every polynomial \( f \in k[x] \) has a root. Equivalently, every polynomial \( f(x) \) factor completely into linear factors:

\[ f(x) = (x - \lambda_1)^{e_1}(x - \lambda_2)^{e_2} \cdots (x - \lambda_n)^{e_n} \]

CONVENTION: In this class we will always assume that \( k \) is algebraically closed.

Algebraic geometry \( / \mathbb{R} \) is very hard.

- Read Chapter 1 of an invitation
- Read Shafarevich 2.1

5/7/01

Warmup

Which

Let \( f \) be a polynomial function \( \mathbb{C}^n \to \mathbb{C} \).

\[ \Gamma = \{ x = y \} \]

\[ (1 + t^2 e^2) W(x^2 - y^2) \leq C^2 \]

\[ x_1, \ldots, x_n \]

\[ W(y^2 - x^2 - xy) \]

\[ W(z^2 - x^2 - y^2) \]

\[ W(xy + xz) \leq A^3 \]

\[ x=0 \text{ or } y = z \]

NOTE: A variety (affine alg. var. of \( \mathbb{A}^n \)) is always closed in the standard complex topology on \( \mathbb{C}^n \)
Hilbert Basis Theorem:

Every affine algebraic variety is defined by finitely many polynomials.

**Example**

\[ V(x^1, y^2 - x^3, z^3 - x^7, y^9 - x^5, \ldots) \leq k^2 \]

\[ (0,0) \subseteq V(x,y) \]

Essentially the same fact: A polynomial ring \( k[x_1, \ldots, x_n] \) (where \( k \) is)

\( \text{a field} \) is

\( \text{Noetherian} \)

Proof of HBT:

\( V = V(\langle yF_1, \ldots, yF_n \rangle) \leq \mathbb{A}_k^n \)

Claim: Let \( I \) denote the ideal of \( k[x_1, \ldots, x_n] \) generated by \( \langle yF_1, \ldots, yF_n \rangle \).

Then \( V = V(I) \)

Because \( \langle yF_1, \ldots, yF_n \rangle \subseteq I \Rightarrow V(I) \supseteq V(\langle yF_1, \ldots, yF_n \rangle) \). Conversely take \( p \in V(\langle yF_1, \ldots, yF_n \rangle) \).

Take any \( g \in I, \) \( \text{nik } g = r, F_1 + \cdots + rF_n \) for some \( r \in k[x_1, \ldots, x_n] \).

\[ p \in \langle yF_1, \ldots, yF_n \rangle \Rightarrow y(F_1) = 0 \]

Since \( I \) is finitely gen, \( I = (F_1, \ldots, F_n) \), \( \text{then } V(F_1) = V(I) = W(F_1, \ldots, F_n) \)

so \( V \) is defined by finitely generated.

**Zariski Topology** on \( \mathbb{A}_k^n \)

**Definition** The \( \mathcal{Z} \)-top on \( \mathbb{A}_k^n \) is that whose open sets are the complements of the affine algebraic subvarieties in \( \mathbb{A}_k^n \).

**Example** On \( \mathbb{A}_1 \) open sets are complements of finite sets of points.

On \( \mathbb{A}_2 \) closed sets are unions of finite collections of points and finitely many "curves".

\[ V(x(y+1), x(y+x^2)) \leq \mathbb{A}_2 \]

\[ (-1,1) \quad (1,1) \]

**Caution** \( \mathcal{Z} \)-top is never Hausdorff (except in trivial case)
There's an induced subspace Zariski topology on every variety. (so closed sets are sub-varieties)

- Open sets are always, always overlap if the variety is irreducible.

2) Finite union of sub-varieties are sub-varieties.

\[ V = V(F_1, \ldots, F_k) \quad \text{and} \quad W = V(F_{k+1}, \ldots, F_s) \]

\[ V \cup W = V(F_1, \ldots, F_k, F_{k+1}, \ldots, F_s) \]

Note: \( \phi \in V(F_i) \iff \phi \notin V(F_i) \]

2) Say \( \phi \in V(F_i, G_j) \) \( \iff \phi \notin V(F_i) \)

MORPHISMS OF VARIETIES

What kind of maps \( V \to W \) are in the world of algebraic geometry?

**Definition.** A regular map (or morphism) \( V \to W \) where \( V \) and \( W \) are affine algebraic varieties is the restriction of a polynomial map on the ambient affine spaces.

\[
\begin{align*}
\mathbb{A}^n & \to \mathbb{A}^m \\
(f_1, f_2, \ldots, f_m) & \mapsto (f_1, f_2, \ldots, f_m)
\end{align*}
\]

**Example.**

1. \( \mathbb{A}^1 \to \mathbb{A}^2 \)

\[
t \mapsto (t^2, t^3)
\]

2. \( \mathbb{A}^1 \to \mathbb{A}^2 \), \( V = V(x^2 - y^2) \subset \mathbb{A}^2 \)

\[
t \mapsto (t^2, t^3)
\]

Any linear or even affine change of coordinates in \( \mathbb{A}^n \) is allowed.

\[
(x_1, \ldots, x_n) \mapsto \left( \sum_{j=1}^n \lambda_j x_j \right)
\]

4. Projection: \( \mathbb{A}^2 \to \mathbb{A}^1 \)

\[
(x_1, x_2) \mapsto x_{1/2}
\]

Read: An Invitation: 2.1, 2.2, 2.3, 2.4

Schaffner: 2.2, 2.3 (lightly)
1) Define an isomorphism of varieties. (E.g. map what is even 0 to what is even 0)
2) Are the following isomorphisms?

\[ A^1 \rightarrow C = \{ y - x^2 \} \leq A^2 \quad \text{and} \quad A^2 \rightarrow A^1 \]

\[ (x, y) \mapsto x \quad \text{for } x = y \]

\[ A^1 \rightarrow D = \{ y - x^3 \} \leq A^2 \]

\[ (x, y) \mapsto (x^2, y) \]

Isomorphisms are home in Zariski top.

I. Hilbert Nullstellensatz

Fix a variety \( V \subseteq A^N \). Consider the ideal \( I = \mathfrak{I}(V) \) of all polynomials on \( A^N \) vanishing at all points of \( V \).

\[ \mathfrak{I}(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \ \forall p \in V \} \]

This is an ideal of \( k[x_1, \ldots, x_n] \)

**Easy Fact**

a) \( \mathfrak{I}(V) \) is a radical ideal (DEF: if \( f \) is radical if whenever \( f(x)^2 \) \( \in \mathfrak{I}(V) \) then \( f \) \( \in \mathfrak{I}(V) \)

b) \( \overline{V} = V(\mathfrak{I}(V)) \)

because \( p \in \overline{V} \) if and only if \( f(p) = 0 \) for all \( f \in \mathfrak{I}(V) \).

2) Take \( f \) such that \( f(p) = 0 \) for all \( p \in \overline{V} \)

\[ f \in \mathfrak{I}(V) \]

3) By definition of \( \mathfrak{I}(V) \), \( f(x) \in \mathfrak{I}(V) \)

**Much Harder Fact** = Hilbert Nullstellensatz

Let \( k \) be an algebraically closed field, and let \( I \) be an arbitrary ideal.

Then \( \mathfrak{I}(V(I)) = \{ f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \ \forall p \in V(I) \} \)

Proof of this is easy and doesn't use \( k \)-alg. closure or \( I \)-radical. \( \mathfrak{I}(V(I)) = \mathfrak{I}(V(I)) \)

(because \( f \in \mathfrak{I}(V(I)) \), need to check \( f(p) = 0 \) \( \forall p \in V(I) \) that \( f(p) = 0 \)).

Another way to state the Nullstellensatz:
For $I$ an arbitrary ideal in $k[x_1, \ldots, x_n]$ with $k$ alg. closed

$$\Pi(V(I)) = \text{Rad}(I) = \sqrt{I}$$

Proof (left to the reader)

The Nullstellensatz gives a correspondence between geometry and algebra.

Geometry \hspace{1cm} Algebra

\[ \mathbb{A}^n \hspace{2cm} (a) \in k[x_1, \ldots, x_n] \]

\[ V \hspace{1cm} \Pi(V) \text{ radical ideal} \]

\[ V(I) \hspace{1cm} I \text{ radical ideal} \]

$$V \subseteq W \hspace{2cm} \Pi(V) \supseteq \Pi(W)$$

\[ p = (x_1, \ldots, x_n) \text{ maximal ideal} \]

In $\mathbb{A}^1$:

\[ \{x_1, \ldots, x_n\} \hspace{1cm} (x-\lambda_1)(x-\lambda_2) \cdots (x-\lambda_n) \]

(a polynomial with repeated roots is completely determined by its zeros)

For Nullstellensatz really need $k = \overline{k}$ (alg. closed)

**EXAMPLE** $\mathbb{A}_k^1 \ni V(x^2+y^2) \subseteq \mathbb{R}^2$

\[ (0,0) \]

\[ \Pi(V) = (x,y) = \{ f \in k[x,y] \text{ s.t. homogeneous term of } f \}

\[ (x^2+y^2) = \text{radical ideal} \]

\[ \Pi(V(\Pi(V))) \neq I = \text{Rad} I \]

\[ V(x^2+y^2) = V(x+iy)(x-iy) \subseteq \mathbb{R}^2 \text{ ellipse} \]

**COORDINATE RING**

Fix $V \subseteq \mathbb{A}^n$ an affine variety over $k$.
**Definition** A regular function $\varphi : V \to k$ is a function $V \to k$ which is a restriction of a polynomial function $k^N \to k$.

Equivalently it's a morphism $V \to A^1$.

**Definition** The coordinate ring of $V$, denoted $k[V]$, is the set (ring) of all regular functions on $V$.

Notice there is a natural ring homomorphism $k[X_1, \ldots, X_N] \to k[V]$.

$$f \mapsto f|_V$$

Caution: if $V$ is not $1$-dimensional, in general $f|_V$ is not $1$-dimensional.

**Example** $V = V(x^2 + y^2 - z^2) \subset A^3$

$F = 1$ and $G = x^2 + y^2 - z^2 + 1$

1. $\ker(p) = \mathbb{V}(V)$

So $k[V] \cong k[x_1, \ldots, x_N]/\mathbb{V}(V)$

Read: Invit. 2.5.2.6.

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Warmup: Coordinate maps for $A^N$

$$k^N \to k[X_1, \ldots, X_N]$$

$$(x_1, \ldots, x_N) \mapsto k[X_1^2, \ldots, X_N^2]$$

A line in $A^N$ $k[x_1, \ldots, x_N]$ is $k[1]$

$$V((y-x^2)^2) \subset A^2 \quad k[x_1, y]$$

$$k[x_1, y]/(y-x^2)^2$$

radical

Q. For $V \subset A^N$, what is the corresponding object in $k[V]$?

For a pt in $V$, $(p) \cong \mathbb{V}(V)$ (maximal ideal)

A subvariety of $V$ containing $\mathbb{V}(V)$ is a set $V$ (incl. $\mathbb{V}(V)$) $p \in V \subset A^N$

$$V((x+y+c))^2 \subset A^2$$

$$k[x_1]/(x+y+c) = k[t]$$

$$t \mapsto \frac{x+y+c}{6}$$

$$x \mapsto t - \frac{c}{6}$$
Recall equivalence of categories between affine varieties & algebras.

**Definition** The coordinate ring of \( V \subseteq \mathbb{A}^n \) is the set of polynomials on \( \mathbb{A}^n \) restricted to \( V \).

\[
\mathbb{R}[x, \ldots, x_n] \xrightarrow{\pi(V)} \mathbb{R}[[V]]
\]

Suppose \( V \xrightarrow{f} W \) is a regular map. Then there is a natural map \( \mathbb{R}[W] \xrightarrow{f^*} \mathbb{R}[V] \)

\[
g \mapsto g \circ f
\]

called pullback (or fiberwise restriction).

This map is a map of \( \mathbb{R} \)-algebras.

**Example**

\[
\mathbb{A}^3 \xrightarrow{\phi} \mathbb{A}^2
\]

\[
(x, y, z) \mapsto (x^2 + y^2 + x, y^2 + xy - z^2)
\]

\[
\mathbb{R}[\mathbb{A}^3] \xrightarrow{\phi^*} \mathbb{R}[\mathbb{A}^2]
\]

\[
\{ t \mathbb{R}[V] \mapsto \mathbb{R}[\mathbb{A}^2] \}
\]

\[
f_{\phi} = \left( x^2 + y^2, y^2 + xy - z^2 \right)
\]

\[
\mathbb{R}[x, y, z] \xrightarrow{f^*} \mathbb{R}[u, v]
\]

\[
x^2 + y^2 \mapsto u
\]

\[
y^2 + xy - z^2 \mapsto v
\]

\[
F^*(u, v) = (x^2 + y^2)(y^2 + xy - z^2)
\]

**Theorem** (Equivalence of categories)

Fix a \( \mathbb{R} \)-algebra field.

(i) Every finitely generated reduced \( \mathbb{R} \)-algebra is isomorphic to the coordinate ring of some affine algebraic variety.

(ii) Every \( \mathbb{R} \)-algebra homomorphism between such \( \mathbb{R} \)-algebras is the pullback of some regular map between the corresponding varieties.

In other words, there's a category (anti-)equivalence between affine varieties and finitely generated reduced \( \mathbb{R} \)-algebras.

**Definition** A ring is reduced if it has no nilpotent elements.
EXERCISE If $I \subseteq A$ is an ideal then $A/I$ is reduced $\iff I$ is radical.

Proof of the theorem:

(i) Say $R$ is a $k$-algebra generated $\Theta_1, \ldots, \Theta_n$,

\[ k[X_1, \ldots, X_n] \xrightarrow{\pi} R \]

$\Theta_i \mapsto \Theta_i$. So $R = k[X_1, \ldots, X_n]/I$ where $I = \text{ker}(\pi)$. 

Because $R$ is reduced, $I$ is radical. Let $V = V(I) \subseteq A^n$. Note $k[V] \subseteq R$.

(ii) Say $R \xrightarrow{\phi} S$ is a $k$-algebra map between finitely generated reduced $k$-algebras. From (i) we have $R = k[X_1, \ldots, X_n]/I(V)$ for some $V \subseteq A^n$.

\[ S = k[Y_1, \ldots, Y_n]/I(W) \quad \text{for some } W \subseteq A^n, \quad \text{where } I = \pi(W), \quad J = \pi(I) \]

$\phi: k[X_1, \ldots, X_n] \xrightarrow{I} k[Y_1, \ldots, Y_n]$.

\[ X_i \mapsto F_i(Y_1, \ldots, Y_n) \]

Caution: We're abusing notation: these are really equivalence classes.

Define a regular map

\[ \begin{array}{ccc} A^m & \xrightarrow{(F_1, \ldots, F_n)} & A^n \\ U^m & \xrightarrow{U^m} & U^n \\ W & \xrightarrow{F} & V \end{array} \]

Need to check $F(W) \subseteq V$. Take $p \in W$. Want to show $F(p) \in V - W(I)$. Need $F(p) \in J$. 

To check $g(F(p)) = 0$. But $g(F(p)) = g(F(p))$. 

To check, it suffices to show that $g(z) \in J$. Obvious because it's well-defined.

Also note, even though $S$ is an algebra, the map $F$ from $S$ to $F(S)$, is well-defined.

To prove the algebraic assertion, see [coordinatewise].

Read 3.1, 3.2. 

9/14/01  PROJECTIVE SPACE

**Definition**  Fix $V$ any vector space. The projectivized space $P(V)$ is defined as the set of 1-dimensional vector subspaces of $V$.

**Example**  $P^1_{\mathbb{R}} = P(\mathbb{R}^2)$

(Notation: $V = k^n$ where $P(V)$ is also denoted $P^n_k$ or $P^n$ where $k$ is understood)

$S_0: \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$

$P^2_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^3)$

$\mathbb{P}^2 = \mathbb{P}(\mathbb{R}^3)$

$P^2_{\mathbb{R}} = \mathbb{A}^2_{\mathbb{R}} \cup P^1$

$P^1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2)$ - Riemann sphere

$P^n = \mathbb{P}(\mathbb{K}^{n+1}) = \mathbb{A}^n_{\mathbb{K}} \cup \mathbb{A}^{n-1}_{\mathbb{K}} \cup \mathbb{A}^1_{\mathbb{K}} \cup \mathbb{A}^{0}_{\mathbb{K}}$

**Note**  $P^n = \mathbb{A}^n_{\mathbb{K}} \cup \mathbb{A}^{n-1}_{\mathbb{K}}$

There are no such things as parallel lines in the projective plane. (all lines intersect!)
**BASIC FACT** \( P^n \) has a natural cover of by \( (n+1) \) copies of \( \mathbb{A}^n \)

\[ P^n = \mathbb{A}^n \cup \ldots \cup \mathbb{A}^n \] (not disjoint)

**EXAMPLE** \( P^1 = \mathbb{A}^1 \cup \{0\} \)

\[ P^2 = \mathbb{A}^2 \cup \{0\} \cup \{0\} \]

---

Homogeneous coordinates for \( P^N \)

Each point in \( P^N \) is a 1-dim subspace of \( k^{N+1} \). Represent \( p \in P^N \) by choosing any basis for it

\[ p = [\lambda_0: \lambda_1: \ldots : \lambda_N] \] where \( (\lambda_0, \lambda_1, \ldots, \lambda_n) \) is basis for the 1-dim subspace \( p \)

So... \( P^N = k^{N+1} \setminus \{0\} \) where \( (\lambda_0, \lambda_1, \ldots, \lambda_N) \sim (\mu_0, \mu_1, \ldots, \mu_N) \) if they lie on the same line through 0 in \( k^{N+1} \)

\[ \lambda_i \in k \setminus \{0\}, \mu_i \in k \setminus \{0\}, \lambda_i = \mu_i \] for \( i = 0, \ldots, N \)

Now... \( U_0 \subset P^N \) \([\lambda_0: \lambda_1: \ldots : \lambda_N] \mid \lambda_0 \neq 0\)

Next... \( U_0 \subset \mathbb{A}^N \)

\[ [\lambda_0, \lambda_1, \ldots, \lambda_N] \rightarrow [\frac{\lambda_0}{\lambda_0}, \frac{\lambda_1}{\lambda_0}, \ldots, \frac{\lambda_N}{\lambda_0}] \]

\[ [\mu_0, \mu_1, \ldots, \mu_N] \rightarrow [\frac{\mu_0}{\mu_0}, \frac{\mu_1}{\mu_0}, \ldots, \frac{\mu_N}{\mu_0}] \]

Clearly \( P^N = \bigcup_{i=0}^{N} U_i \) (standard affine cover of \( P^N \)) or affine charts.

Homogeneous coordinates of \( P^n \) \( k[x_0, \ldots, x_n] \) is really...
CAUTION $f \in k[x_0, \ldots, x_N]$ is never a function on $\mathbb{P}^N$ unless $f = \text{const.}$

**Example**

$f = x_0^2 + x_1 x_2 \in k[x_0, x_1, x_2]$

\[
f(a, b, c) = a^2 + b c^2
\]
\[
f(c, d, 2e) = c^2 + e c^2 = (impossible)
\]

9/18/01

**PROJECTIVE VARIETIES**

**Warning:**

1) $k$ algebraically closed $\Rightarrow k$ is infinite
2) I gave $\mathbb{A}^n$ radical

In $\mathbb{P}^n$ we have homogeneous coordinates $[x_0 : x_1 : \ldots : x_N]$

**Definition** The homogeneous coordinate ring $k[x_0, \ldots, x_N]$. Cautions: The elements in $k[x_0, \ldots, x_N]$ are not functions on $\mathbb{P}^n$ except the constant ones.

**Definition** A polynomial is homogeneous of degree $d$ if each of its terms is a monomial of degree $d$, i.e.

\[
\sum_{\mathbf{a} \in \mathbb{N}^n} \lambda_{\mathbf{a}} x_0^{a_0} \ldots x_N^{a_N} = \sum_{\mathbf{a} \in \mathbb{N}^n} \lambda_{\mathbf{a}} x_0^{a_0} \ldots x_N^{a_N}
\]

**Fact** If $f \in k[x_0, \ldots, x_N]$ homogeneous of degree $d$, then

\[
f(\lambda x_0, \ldots, \lambda x_N) = \lambda^d f(x_0, \ldots, x_N)
\]

**Proof:** Enough to check for monomials $f(x_0, \ldots, x_N) = x_0^{a_0} \ldots x_N^{a_N}$

\[
f(\lambda x_0, \ldots, \lambda x_N) = \lambda^d f(x_0, \ldots, x_N)
\]

**Consequence:** $f \in k[x_0, \ldots, x_N]$ is a well-defined subset of $\mathbb{P}^N$ iff every $(\lambda a_0, \ldots, \lambda a_N) \in V(f)$ for any $\lambda \in k$.

So $V(f)$ is a well-defined subset of $\mathbb{P}^N$. 
DEFINITION A projective algebraic variety is the common zero set of a collection of homogeneous polynomials in $k[x_0, x_1, \ldots, x_n]$. 

$$V = V\left(\{f_1, f_2, \ldots, f_l\} \mid x_0 \neq 0\right) \subseteq \mathbb{P}^n$$

**REMARK/DEFINITION** The affine algebraic variety in $\mathbb{A}^n$ defined by the same polynomials is called the affine cone over $V$.

**NOTE** From what we've already shown about affine varieties:

1. $V \subseteq \mathbb{P}^n$ is actually defined by finitely many homogeneous polynomials.

$$V(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$$

$A^2 = \{ (0, 0, w) \mid w \neq 0 \}$

$V(x, y, z) \subseteq \mathbb{P}^2$

$A^2 = \{ (x, y, z) \mid x \neq 0 \}$

$V(x, y, z) \subseteq \mathbb{P}^2$

$V \cap U_x \subseteq U_x \subseteq A^2$

$V(W(x^2 - y^2)) \cap U_x \cap U_y$

2 points at $\infty$ 

$V \cap U_x \cap U_y$ has 2 points at $\infty$.

$\mathbb{P}^2 \setminus U_x \setminus U_y = V(+) \subseteq \mathbb{P}^2$

$V \cap U_2: \{ [1 : 0 : 0], [0 : 1 : 0] \}$

**DEFINITION** The Zariski topology on $\mathbb{P}^n$ is that whose closed sets are the projective subvarieties.

**PROJECTIVE VARIETIES NULLSTELLENSATZ:**
The projective varieties in $\mathbb{P}^n$

1-1

Homogeneous radical ideals in $k[[x_0, \ldots, x_n]]$ except $(x_0, x_1, \ldots, x_n)$

An ideal is homogeneous if it can be generated by homogeneous elts.

$V(I) = \bigcap_{f \in I} \{ x \in \mathbb{A}^n | f(x) = 0 \}$

Homogeneous, we need $x_0$.

Read 3.1, 3.2, 3.3, 3.4, 3.5

9/19/01

Warning: Thinking of $\mathbb{A}^2 \rightarrow \mathbb{P}^2$ via $(x, y) \mapsto [x : y : 1]$. What are the points at $\infty$ for the varieties:

1. $W(x-y-1) \subseteq \mathbb{A}^2$
   $xy - z = 0 \text{ at } \infty$, $x = 0, y = 0$

2. $W(x - y) \subseteq \mathbb{A}^2$
   $[x : y : 1]$
   $x = y \text{ at } \infty$, $x = 0, y = 0$

So, $y = 0$ or $z = 0$

The two points:

$[0:1:0] \text{ and } [1:0:0]$

2 points: $[0:1:0] \text{ and } [1:0:0]$

1 point: $[0:0:0]$

I. PROJECTIVE CLOSURE (a way of "compactifying" in the world of algebraic geometry)

Let $V \subseteq \mathbb{A}^n$ be affine variety. Think of $\mathbb{A}^n$ as sitting in $\mathbb{P}^n$ in the usual way.

1 of the affine plane $\mathbb{C}^2$.

$[x_1, \ldots, x_n] \mapsto [x_1 : \ldots : x_n]$

Definition: The projective closure of $V$ is the closure of $V$ in the Zariski topology on $\mathbb{P}^n$. This is a projective variety $\overline{V}$ containing $V$. 

$\mathbb{P}^n$ denotes $\mathbb{A}^n \cup \{ \infty \}$.
\( V = W(\{x, y\} \cup \{z\} \cup \{w\}) \)

\( \nabla \cap (\mathbb{P}^2 - \mathbb{A}_x^2) = \nabla \cap W(z) = W(x^2 - z, z) = W(xy, z) = \{(0, 0, 1), (0, 1, 0)\} \)

\( V = W(x^2 - y), \quad \nabla = W(x^2 - y, z) \leq \mathbb{P}^2 \)

\( \nabla \cap (\mathbb{P}^2 - \mathbb{A}_x^2) = \nabla \cap W(z) = W(x^2, z) = \emptyset \{(0, 1, 0)\} \)

**Definition:** Given a polynomial \( f \in k[x_1, \ldots, x_n] \) of degree \( d \) (not necessarily homogeneous), its homogenization is the unique homogeneous polynomial \( F \) of degree \( d \) in \( k[x_0, x_1, \ldots, x_n] \) such that \( F(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \).

**Construction:** Write \( f = f_0 + f_1 x_1 + \cdots + f_d x_1^d \). Then \( F = f_0 x_0^d + f_1 x_0^{d-1} x_1 + \cdots + f_d x_1^d \).

**Caution:** If \( V = W(f, f_1, \ldots, f_d) \leq \mathbb{A}^n \) then the projective closure is not necessarily simply \( W(F_1, \ldots, F_d) \)

\( \mathbb{Z}^n \rightarrow \mathbb{C} \times \mathbb{C} \times \mathbb{C} \)

**Example:** Twisted cube: \( \{(t, t^2, t^3) | t \in \mathbb{A}^1 \} \leq \mathbb{A}^3 \leq \mathbb{P}^3 \)

\( V(z^2 - x^3, y^2 - x^3) \)

\( V(z^3 - x^3, y^2 - x^3) \)

Point at \( 0 \): \( [0: 0: 0: 1] \)

\( W = V(2w^2 - x^3, 2y - x^3) \leq \mathbb{P}^3 \)

Caution: \( W \) is not \( \nabla \)

1. \( W \cap W = V(z^2 - x^3, y^2 - x^3) \leq \mathbb{A}^3 \)

2. \( W \cap (\mathbb{P}^3 - \mathbb{A}_w^2) = W \cap W(W) = W(2w^2 - x^3, 4y - x^3, w) = W(x^3, x^3, w) = \)

\( = V(x, w) = \{(0, 0, 1, z) \} \leq \mathbb{P}^3 \) (im)
so we may assume \( \mathcal{V} \) is given encoded in \( V \).

Try \( \mathcal{V}(z-x^3, y-x^2) = \mathcal{V}(z-x^3, y-x^2, x^2-y^3) \)

Try a new \( \mathcal{V} = \mathcal{V}(w^2 z - x^3, w y - x^2, x^2 - y^2) \)

\[ W \cap U_2 = \{ 0 \} \]

\[ W \cap W(U) = \mathcal{V}(w, x^2, x^2, y^2) \]

**Theorem** Let \( \mathcal{V} \subseteq \mathbb{A}^n \) be an affine variety and let \( \mathcal{V} \subseteq \mathbb{P}^n \) be its projective closure. Then if \( \mathcal{I} = \mathcal{I}(\mathcal{V}) \subseteq k[x_1, \ldots, x_n] \), then the ideal \( \mathcal{I}(\mathcal{V}) \) is generated by the homogenizations of all the elements of \( \mathcal{I} \).

**Proof** Let \( \mathcal{V} \subseteq \mathbb{A}^n \) generated by the homogenizations of all elements of \( \mathcal{I} \). We want to show \( \mathcal{V} = \mathcal{W}(\mathcal{I}) \) and \( \mathcal{I} \)

\[ \mathcal{V} \subseteq \mathcal{W}(\mathcal{I}) \]

Consider \( f \in \mathcal{W}(\mathcal{I}) \). For \( \mathcal{V} \) to be \( \mathcal{W}(\mathcal{I}) \), \( f \) must vanish at all points of \( \mathcal{V} \).

Therefore, if \( \mathcal{V} \) is projective, \( f \) must vanish at all points of \( \mathcal{V} \).

Conversely, if \( \mathcal{V} \subseteq \mathcal{W}(\mathcal{I}) \), we can show that \( \mathcal{V} = \mathcal{W}(\mathcal{I}) \).

\[ \mathcal{V} \subseteq \mathcal{W}(\mathcal{I}) \]

Indeed, \( \mathcal{I} \) is radical.

**Remark** Let \( \mathcal{V} \) be algebraic, then \( \mathcal{V} \subset \mathcal{W}(\mathcal{I}) \).

9/20/01

Wwwww. Consider the map \( \mathbb{P}^1 \longrightarrow \mathbb{P}^2 \)

\[ [a:b] \longrightarrow [ax^2 + bx + d : ay^2 + cy + d] \]

**Q:** Why well-defined?

**Q:** What does \( \mathcal{I} \) mean?

**Q:** Is \( \mathcal{I} \) 1-1? Weis. How to \( \mathcal{I} \) mean?

\[ \mathcal{I} \langle x^2 + y \rangle \]

\[ \text{In} (x^2 + y) \leq \mathcal{I}(x z - y^2) \]

\[ \text{In} (x^2 + y) \leq \mathcal{I}(x z - y^2) \]

In fact, \( \mathcal{I}(x z - y^2) \)

**Inversion:** \( \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \)

\[ [0:1] \longrightarrow [0:1] \]

\[ [1:0] \longrightarrow [1:0] \]

\[ [x:y] \text{ if } x \neq 0 \]

\[ [0:1] \]
This is an example of a morphism (in fact an isomorphism) of projective varieties (called a "Veronese" map).

**Definition (Projective)**
A morphism between projective varieties \( V \to W \) is a set map with a property that \( V \cap U \) is an affine set open subset \( U \) (in fact such that on \( U \), the map agrees with a polynomial map \( \mathbb{P}^N \to \mathbb{P}^M \)) where \( F \in k[V, \ldots, X_v] \) are homogeneous of the same degree.

**Examples**

\[ \mathbb{P}^1 \to \mathbb{P}^d \]
\[ [s:t] \mapsto [s^2 \ldots s^d : t^2 \ldots t^d] \]
This is called the Veronese map of degree \( d \) on \( \mathbb{P}^1 \).

Let \( V \subseteq \mathbb{P}^d \) be the image set \( V = V(\text{2 \times 2 minors of} \begin{bmatrix} 2 & \cdots & 2 \end{bmatrix}) \)
\[ \text{I} \]

(Observe that \( \text{Im} V \subseteq V(\text{I}) \), other discussion for \( \text{HH} \))

\[ V \to \mathbb{P}^1 \]
\[ [2a_0 : b_0 : c_0] \mapsto \begin{cases} \text{if } a_0 \neq 0 & [a_0 b_0 : c_0] \text{ if } a_0 \neq 0 \\ \text{if } c_0 \neq 0 & [a_0 : b_0 c_0] \text{ if } c_0 \neq 0 \end{cases} \]

Note: if \( [2a_0 : b_0 : c_0] \in V \), then \( 2a_0 \) or \( 2b_0 \)

\[ [2a_0 : b_0 : c_0] = [2a_0 : d^{a_0} : d^{b_0}] = [d^{a_0} d^{b_0} d^{c_0}] = [d^{2a_0} : d^{2b_0} : d^{2c_0}] = (2a_0, 2b_0, 2c_0) \]
This shows that \( V \cong \mathbb{P}^1 \). This is called the twisted "d-ic" or a...
\[ \mathbb{P}^n \rightarrow \mathbb{P}^n \]
\[ [x_0, \ldots, x_n] \rightarrow [a_0, x_0, \ldots, a_n x_n] \]

\[(a_{ij}) \text{ is a } (n+1) \times (n+1) \text{ matrix. This is a linear map of the vector } k^{n+1} \]
\[ \text{with kernel } \mathbb{V} \]

\[ \mathbb{P}^n \rightarrow \mathbb{P}^n \]
\[ [x_0, \ldots, x_n] \rightarrow [x_0, \ldots, x_n] \text{ (not well-defined)} \]

These are called projective changes of coordinates.

**FACT** There are the only automorphisms of \( \mathbb{P}^n \) (rigidly non-obvious).

For \( \mathbb{P}^1 \), \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \)
\[ [x : y] \rightarrow [a x + b : c y + d] \]
\[ [x : y] \rightarrow \frac{ax + b}{cx + d} \text{ with } ad - bc \neq 0 \]
\[ x \rightarrow \frac{ax + b}{cx + d} \]

Intuitively, a morphism from a projective variety is the same thing as a collection of regular maps from each of its affine patches.

**EXAMPLE** The Veronese maps on \( \mathbb{P}^n \).

Fix \( d > 1 \), \( \mathbb{P}^n \rightarrow \mathbb{P}^{n(d+1)} \)
\[ [x_0, \ldots, x_n] \rightarrow [x_0^d, x_0^{d-1} x_1, \ldots] \]

**FACT** # of degree \( d \) monomials in \( n+1 \) variables is \( \binom{d+n}{n} \)

Put down \( d \times d \) "slots" \[ \begin{array}{c}
\end{array} \]
Choose all of them. This

\[ \begin{array}{c}
\end{array} \]
\[ \begin{array}{c}
\end{array} \]
\[ \begin{array}{c}
\end{array} \]

gives us \( N! \) choices. This

\[ \begin{array}{c}
\end{array} \]
\[ \begin{array}{c}
\end{array} \]
\[ \begin{array}{c}
\end{array} \]

So in our example \( M = \binom{d+n}{n} - 1 \)
This is well-defined.
It's an isomorphism onto its image. So, the inverse map is given as follows.

Consider the Veronese map

$$\mathbb{P}^2 \rightarrow \mathbb{P}^5$$

$$[x:y:z] \mapsto [x^e \cdot x^2 \cdot y^2 \cdot z^2]$$

What is the image of the conic $$W(x^2 + y^2)$$?

What is the image of an arbitrary conic $$W(ax^2 + bx + cy^2 + dz^2 + efz + fgz + fhz + gjz)$$?

Image of $$P_2 = V_0 \in \mathbb{P}^5$$

$$\mathbb{P}^2 \rightarrow V \in \mathbb{P}^5$$

$$W(x^2) \rightarrow W(8-1)$$

Define

$$V \cap H$$, where $$H$$ is any hyperplane (i.e., $$W(1)$$ is a linear form) is called a "hyperplane section of $$V".$$
SEGRE MAPS

1. $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ (as sets)
   $\mathbb{P}^n \times \mathbb{P}^n \neq \mathbb{P}^{2n}$ (as sets)

Purpose of Segre Map $\sigma_{n,m}$ is to endow the set $\mathbb{P}^n \times \mathbb{P}^m$ with a structure of a projective variety.

Case: $\mathbb{P}^1 \times \mathbb{P}^1$

$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma_n} \mathbb{P}^3$

$[(s: t), [u:v]] \rightarrow [su: sv: tu: tv]$

$x/y = u/w$ well-defined unless $x = 0$, $w = 0$.

Image $\sigma_n = \mathbb{V}(xw-yZ)$

$\mathbb{V}(xw-yZ) = \mathbb{V}(xw-yZ)$

$\Rightarrow \alpha = bc$, $\beta = \mathbb{V}(bc \cdot c \cdot 1) = \mathbb{V}(bc, \alpha, 1)$

$\sigma_{n,m} = \mathbb{V}((2 \times 2)_{\beta \alpha})$

Let $\Sigma_{n,m} = \mathbb{V}(xw-yZ)$, is $\Sigma_{n,m}$ surjective? Yes, the inverse:

$\mathbb{P}^3 \xrightarrow{\sigma_{n,m}} \mathbb{P}^1 \times \mathbb{P}^1$

$[xy: z: w] \rightarrow [(x:y), [x:w]]$ if $x \neq 0$

$[(x:y), [y:w]]$ if $y \neq 0$

$[(z:w), [x:y]]$ if $z \neq 0$

$\sigma_{n,m}^{-1}$ is well-defined because it takes the same values on the overlap.

This identifies the set $\mathbb{P}^1 \times \mathbb{P}^1$ with a subvariety of $\mathbb{P}^3$, namely $\mathbb{V}(xw-yZ)$

$= \mathbb{P}(\text{all rank 2 x 2 matrices of rank } 1)$

Local Picture:

$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma_{n,m}} \mathbb{P}^3$

$[u: v] \rightarrow [u: v: x: y]$

$\mathbb{A}^2 \rightarrow \mathbb{A}^3$
\[ \begin{bmatrix} 1 & t \\ 1 & v \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & t & v \\ v & z \end{bmatrix}} \Sigma \left( W, \delta^2 \right) \]

Think about a "grid" line.

\[ \begin{bmatrix} A' \times A' \end{bmatrix} \]

For a pt. \( P \):

\[ \left( P \times A' \right) \xrightarrow{\begin{bmatrix} 1 & 6 & 6, v \end{bmatrix}} \begin{bmatrix} v & 0 & 0 \\ 0 & v & 0 \end{bmatrix} \]

Image of \( P \times A' \) in \( A' \) is the actual line \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} | v \in k \right\} \).

General case.

\[ \begin{array}{c}
P^m \times P^m \\
\xrightarrow{\begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}} P^{(m)(m-1)} - 1
\end{array} \]

\[ \sum_{ij} \begin{bmatrix} x_i, x_j \\ y_i, y_j \end{bmatrix} \rightarrow \begin{bmatrix} x_i \\ y_i \end{bmatrix} \]

**Proposition.** The image \( \sum_{mm} \) of \( \sigma_{mm} \) is a closed subvariety of \( P^{(m)(m+1)/2} \) given by

\[ W(2 \times 2(z_{ij})) \quad Z_{ij} = Z_{ji} \]

Proof. TFAE for \( (m)(m+1)/2 \) and \( A \)

1. Rank \( A \leq t \)
2. All \( (t+1) \times (t+1) \) minors vanish
3. \( A \) factors on \( B \) where \( B \) is \( (t+1) \times t \) and \( C \) is \( t \times (m+1) \)

There are natural projections

\[ P^{(m+1)(m+1)/2 - 1} \xrightarrow{\sum_{mm}} P^n \]

\[ \sum_{mm} \xrightarrow{\pi_m} P^m \]

\[ \sum_{mm} \xrightarrow{\pi_m} P^m \]

**Moral.** The varieties \( \Sigma_{mm} \) are considered by algebraic geometers to be the \( m \times m \) subvarieties of \( A \).
A quasi-projective variety is an open subset of a closed subset of $P^n$. Equivalently, an intersection of an open & closed set in $P^n$.

**Examples**

1. Every projective variety is quasi-projective.
2. Every affine is quasi-projective $V = A^n \subseteq P^n$

$$V = \overline{V \cap U_0} = \{ (x_0, \ldots, x_n) | x_0 \neq 0 \}$$

3. Open subset of q.p. varieties are q.p.

**Rational Functions:**

Consider $P^n$. Remember, a polynomial $F \in k[x_0, \ldots, x_n]$ does not determine a function on $P^n$ even if $F$ is homogeneous.

But $f = \frac{P}{Q}$ where $P$ and $Q$ are homogeneous of the same degree does define a "function" on $P^n$.

Note: $f$ determines a well-defined function on the open set $U : P^n \setminus V(Q)$, $U \xrightarrow{f} k$.

**Definition** A rational function on $P^n$ is an expression of the form $f = \frac{P}{Q}$ where $P$ and $Q$ are homogeneous of the same degree in $k[x_0, \ldots, x_n]$, where $2$ such expressions are considered equivalent if

1. They determine the same elt of $k(x_0, \ldots, x_n)$

or equivalently

2. If they agree on the open subset of $P^n$ where both are defined.

**Example** On $P^3$, coordinates are $x, y, z, w$

$$f = \frac{x^2 y}{w^2}, \quad g = \frac{x^2 y^2}{x w^2}$$

Equal because they agree on open subset $P^3 \setminus V(z)$. 

**Example** On $P^3$, coordinates are $x, y, z, w$
**Definition** Let \( X \subseteq \mathbb{P}^n \) be a q.p. variety, a rational function on \( X \) is the restriction of a rational function \( f = \frac{P}{Q} \) on \( \mathbb{P}^n \), where we assume that \( Q \) does not simultaneously vanish along any component of \( X \).

Actually, usually algebraic geometers define rational functions only on irreducible varieties.

**Definition** A q.p. variety \( X \) is irreducible if it cannot be written as \( X = X_1 \cup X_2 \) where \( X_1 \) and \( X_2 \) are both closed and proper.

**Example** Let \( V \subseteq \mathbb{A}^n \) be an affine variety, considered as a quasi-projective variety in \( \mathbb{P}^n \), i.e., \( V = \overline{V} \cap U_0 = \mathbb{A}^n \supseteq \mathbb{P}^n \).

**Fact** The rational functions on \( V \) can be identified with the elements of the fraction field of \( k[V] \).

**Proof**: \( f = \frac{P}{Q} \), \( P, Q \) are homogeneous, \( P, Q \in k[x_0, \ldots, x_n] \).

\[
f \bigg|_{U_0} = \frac{P(1, x_1, \ldots, x_n)}{Q(1, x_1, \ldots, x_n)} = f
\]

\[
f \mid_V = \frac{P_1}{Q_1} \quad \text{where } p_1, q_1 \in k[V] \quad \text{and } q_1 \neq 0.
\]

**Example** \( \mathbb{A}^2 \subseteq \mathbb{P}^2 \) \( \bar{X} = \mathbb{A}^2 \):

\[
\frac{x^2 - 2}{x^3 + 1} \in k(V)
\]

**Definition** Let \( X \subseteq \mathbb{P}^n \) be a q.p. variety. A function \( X \to k \)

is said to be a regular function if it is a rational function on \( X \).

**Definition** Let \( X \subseteq \mathbb{P}^n \) be a q.p. variety. A function \( X \to k \) is regular if \( f = \frac{P}{Q} \) with \( Q(x) \neq 0 \).
if $\forall x \in X \exists$ a rational function $\frac{p}{q}$ which is regular at $x$ and such
that $f$ agrees with $\frac{p}{q}$ on some open nbhd of $x \in X$.

(NOTE: $p$ and $q$ may depend on $x \in X$)

**THEOREM** Let $V \subseteq A^n$ be an affine variety, considered as a g.p. variety. Then the regular functions on $V$ (as just defined)
Then $V \rightarrow k$ is a regular function on $V$ (as just defined) if
if $f$ agrees with the restriction of a polynomial function on $A^n$.

9/28/01

This weekend:

Surf: Chap. 2, Chap. 3, 1, 3, 2, Ch. 4, 4, 1, 4, 2, 4, 3, Ch. 5.1

Let $V \subseteq A^n$ be an affine variety (i.e., a Zariski closed set).
Let $\mathcal{O}_V(V)$ be the ring of regular functions on $V$ as defined last time.
($f : V \rightarrow k$ is in $\mathcal{O}_V(V)$ if $\forall x \in V \exists p, q \in k[V], s.t. \frac{p}{q} \in k$ on some open nbhd $U$ of $x$).

$\mathcal{O}(V) =$ regular $f(x) = k[V] =$ coordinate rings.

**THEOREM** For $V$ affine, $\mathcal{O}_V(V) = k[V] =$ ring of poly functions on $A^n \times V$.

Proof: one inclusion is obvious $k[V] \subseteq \mathcal{O}_V(V)$

For the reverse inclusion:

Take $f \in \mathcal{O}_V(V)$

by def. $\forall x \in V \exists p, q \in k[V], s.t. f = \frac{p}{q}$ on some open set $U$ containing $x$.

Note: On $U_x$, $f$ has an expression of the form $g \cdot f = p$, and $x$.
In this case, we can assume $q \cdot f = p$ on all of $U$.
To see union, let $g \in k[V]$ be a function which doesn't vanish at $x$.
But which vanishes everywhere on $U \subseteq \text{closed $f$-set}$.

[can take any $g \in \mathcal{I}(V)$ that doesn't vanish at $x$. If all $g \in \mathcal{I}(W)$ vanish at $x$, then $x \not\in \text{Var}(I(W); W)$]

by replacing $px$ and $q$ by $g \cdot px$ and $g \cdot q$. Now $g \cdot f = g \cdot \frac{px}{q}$ holds.
II. REGULAR MAPS OF QUASI-PROJECTIVE VARIETIES

DEFINITION. A map \((\phi : X \to \mathbb{A}^n)\), where \(X \subseteq \mathbb{P}^n\) and \(\mathbb{A}^n\) are quasi-projective varieties, is regular if each of the coordinate functions \(\phi_i\) is a regular function on \(X\).

EXAMPLE. The map \(\phi : \mathbb{A}^1 \to \mathbb{A}^2\) defined by \(\phi(t) = (t, t^2)\) is regular.
2. $X \rightarrow \mathbb{A}^N$ regular, $\mathbb{A}^N$-affine (Z.-closed under map $f^{\#}$)

The new def. is the same as the old one by the theorem proved.

**NEW & IMPROVED DEFINITION OF "AFFINE VARIETY"**

A quasi-projective variety $X$ is said to be affine if it is isomorphic (as a q.p. variety) to a Zariski-closed subset of $\mathbb{A}^N$ for some $N$.

**EXAMPLES**

1. All Zariski-closed subsets of $\mathbb{A}^N$ are affine.
2. $\mathbb{A}^1 - 0$ is isomorphic to $\{x(0y - 1) \in \mathbb{A}^1 \mid x \neq 0, y = 0\} \cong \mathbb{A}^1$.

In general, let $V$ be any Zariski-closed set in $\mathbb{A}^N$. Let $f \in \mathbb{K}[V]$.

Let $U_f = V \setminus V(f)$. The sets $U_f$ are open affine neighborhoods of $V$.

**IMPORTANT PRINCIPLE**: Every affine variety has a basis of open affine neighborhoods (Exercise).

**EXAMPLE**: $\mathbb{A}^2 - 0$ is not affine (conditions in $\mathbb{P}^2$, i.e., $x, y 
eq 0$).

**WARMUP**

Let $V \subseteq \mathbb{A}^N$ be an irreducible closed subset of $\mathbb{A}^N$.

Let $f \in \mathbb{K}[V]$ be a non-constant section.

Let $U_f = V \setminus V(f)$.

Consider the map: $U_f \rightarrow \mathbb{A}^N \times \mathbb{A}^N$,

$x \mapsto (\frac{x}{f}, \frac{y}{f})$.

Is this surjective? What is its image? Is it "effective", an open inclusion?

Let $f$ be irreducible over its image.

Say $V = V(g_1, \ldots, g_l) \subseteq \mathbb{A}^N$, $g_i \in \mathbb{K}[x_1, \ldots, x_N]$

$\text{Max}^* = \{(\lambda_1, \ldots, \lambda_l) \mid \lambda_i \in V, \lambda_i \neq 0 \text{ for } i = 1, \ldots, l \}$

$\mathbb{V} = \{x \in \mathbb{A}^N \mid f(x_1, \ldots, x_N) = 0\} \subseteq \mathbb{A}^N$.
REGULAR FUNCTIONS on \( U_V = V \setminus W \) where \( W = \{ (x, y, z) : z = 0 \} \) lies.

\[
\mathcal{O}_{U_V}(U_V) = k[V][\frac{1}{t}] = \{ \frac{g}{t^g} : g \in k[V], t \in \mathbb{N} \}
\]

Identify \( U_V \) with \( W \in \mathbb{A}^{n+1} \).

\[
k[w] = \frac{k[x_1, \ldots, x_n, y]}{(g, y)}
\]

\[
\text{Note: There is a natural \( k \)-algebra map} \quad \frac{k[x_1, \ldots, x_n, y]}{(g, y)} \rightarrow \frac{k[x_1, \ldots, x_n]}{(g)}
\]

\[
x \mapsto \frac{x}{y}
\]

Moral: The two ways of defining regular functions on \( U_V \subseteq \mathbb{V} \)-subvarieties agree.

\[
\text{COROLLARY} \quad \text{Every} \; q.f. \; \text{variety has an open affine cover by affine varieties.}
\]

\[
\text{Example} \quad A^2 - \{(0,0)\} = U_x \cup U_y
\]

\[
W \subseteq \mathbb{P}^n \times \mathbb{A}^m \quad W \text{ is a common zero set of polynomials in } x_1, x_2, y_1, \ldots, y_m \text{ that are homogeneous in } x_i \text{ and homogeneous in } y_j
\]

\[
\text{Closed set in } \mathbb{P}^n \times \mathbb{A}^m \text{ is defined by polynomials in } k[x_1, x_2, y_1, \ldots, y_m]
\]

\[
W(x_0^2 y_1 + (x_0^2 + x_0 x_1) y_2) \subseteq \mathbb{P}^3 \times \mathbb{A}^2
\]

\[
\text{Closed set in } \mathbb{P}^n \times \mathbb{A}^m \text{ is defined by polynomials in } k[x_1, x_2, \ldots, y_m]
\]

\[
W(x_0^3 y_1 + x_1 x_2^2 y_2^5) \subseteq \mathbb{P}^n \times \mathbb{A}^m
\]

Compare & contrast: affine \& projective varieties.

\[
\text{Let } V \text{ be affine, and consider a regular map } V \rightarrow \mathbb{P}^n
\]

\[
\text{Is it closed?}
\]

\[
\text{Example: } A^2 \rightarrow A^1 \quad W(xy, 1) \rightarrow A^1 - \text{not closed anymore}
\]
**THEOREM** The image of projective variety under a regular map \((f: X \to Y)\) is closed. Precisely, if \(f: X \to Y\) is a regular map of \(X, Y\), \(V \subseteq X\) is a projective subvariety, then \(f(V) \subseteq Y\) is closed in \(Y\). (and \(\subseteq \mathbb{P}^n\))

**COROLLARY** If \(f: X \to Y\) is a regular map with \(X\) projective, then \(f\) is close.

Direct contrast to "affine case".

**COROLLARY** Let \(V\) be an irreducible \(\mathbb{P}^n\) projective variety and let \(f: V \to \mathbb{P}^1\) be a regular function on \(V\). Then \(f\) is a constant function.

**Proof**:

\[ x \mapsto f(x) \]

As \(V\) is projective, the image is closed in \(\mathbb{P}^1\), so must be a finite set of points. In fact, image of \(f\) is \(\mathbb{P}^0\) \(\to\) a point, hence \(V\) irreducible. \(\Box\)

Examples: Big maps are continuous in \(\mathbb{P}^1\).

**COROLLARY** A regular map \(X \to Y\) where \(X\) is irreducible & projective and \(Y\) a affine must be a map to a point.

**Proof**:

\[ X \to Y \subseteq \mathbb{A}^n \]

Then the coordinate functions of \(X \to (y_1(x), \ldots, y_n(x))\)

are constants.

**EXAMPLE of REGULAR MAP of PROJ VARIETY**

Let \(V \subseteq \mathbb{P}^n\) be a projective variety. Take \(p \in \mathbb{P}^n - V\). Fix a hyperplane \(\mathbb{P}^{n-1} = H \subseteq \mathbb{P}^n\) not containing \(p\).

Define projection from \(p\) to \(H\) as follows:

\(f: V \to \mathbb{A}^n\), look at \((x_1, \ldots, x_n)\) \(f\) intersects \(H\) in a unique pt \(\Pi(f)\) (Bézout).
Note: This extends to a map $P^N - \rightarrow P^{N-1}$ but not well-defined at $p$ itself.

To see it clearly, choose a coordinate for $P^N$ so that $p=[0: \ldots : 0:1]$ is the origin in affine patch $U_0=\{x_0 \neq 0\}$ and $H=V(x_0) \cong P^{N-1} \subset P^N$.

Then $\pi: P^N \rightarrow P^{N-1}$

\[ [x_0: \ldots : x_N] \rightarrow [x_0: \ldots : x_N] \quad \text{(well-defined except at } p) \]

determines a function $\pi^*: \mathcal{O}_{P^{N-1}} \rightarrow \mathcal{O}_{P^N}$.

10/3/01

**Theorem.** Let $V$ be a projective variety and $V \rightarrow P^n$ a regular map, then $f(V) \subset P^n$ is closed.

**Corollary.** A regular map from a projective variety to a projective variety is a closed map.

**Caution.** Regular maps from affine varieties are rarely closed.

**Example.** Projection $(\pi: P^N \rightarrow P^{N-1})$ of a variety $V \subset P^N$, projective variety.

Projection: Fix $p \in P^{N-1}$ and a hyperplane $P^{N-1} \subset P^N$.

Let $\pi: V \rightarrow P^{N-1}$ be a projection from $P^N$. 

\[ [x_0: \ldots : x_N] \mapsto [x_0: \ldots : x_N] \quad \text{(well-defined except at } p) \]
Example. Let $C \subseteq \mathbb{P}^3$ be the twisted cubic
\[
C = \{ [s^3 : s^2 t : st : t^3] | ([s : t] \in \mathbb{P}^1) = V(2 \times 2 (x_0 x_1 x_2 x_3)) = V(x_0 x_1 - x_2^2, x_0 x_2 - x_1 x_3, x_1 x_2 - x_3^2) \}
\]

Let $P = [0, 0, 1, 0]$ lie $C$ and let the hyperplane by $H = V(x_1) = \mathbb{P}^2 \subseteq \mathbb{P}^3$
\[
\mathbb{P}^3 \xrightarrow{\pi} \mathbb{P}^2 \quad \text{(not defined at } P) \]
\[
[x_0 : x_1 : x_2 : x_3] \xrightarrow{\pi} [x_1 : x_2 : x_3]
\]
\[
\mu \quad \mu
\]
\[
C \xrightarrow{\pi} \pi(C)
\]

Then we guarantee that $\pi(C) \subseteq \mathbb{P}^2$ is a projective subvariety.

Claim. $\pi(C) = V(xz^2 - y^3) \subseteq \mathbb{P}^2$.
\[
\pi(C) = \{ [s^3 : st^2 : t^3] | ([s : t] \in \mathbb{A}^2) = V(xz^2 - y^3) \subseteq \mathbb{A}^2
\]
\[
\pi(C) \cap \mu = \{ (x, y) \} \subseteq \mathbb{A}^2
\]
\[
V(x - y^3) \quad V(xz^2 - y^3) \cap \mu = V(x - y^3)
\]

Proof of the claim.

1) Proposition. The graph of a regular map $\pi : Y \rightarrow X$ is closed in $X \times Y$.
\[
\{(\pi(y), y) | y \in Y\} = \pi \subseteq X \times Y \text{ in } Z\text{-closed set in } X \times Y
\]

2) Theorem. If $X$ is projective and $Y$ is a projective, then the canonical projection $X = Y = \pi : X \times Y \rightarrow Y$ is a closed map.

Proof of Thm 1. Assume $X$ is projective and $Y$ is.

Given $V \xrightarrow{\pi} \mathbb{P}^n$ regular map with $V$-projective. Look at the guy
Proof of the proposition:

1) WLOG, \( Y \in \mathbb{P}^n \). Let \( X \xrightarrow{\rho} \mathbb{P}^n \), \( \pi: \mathbb{P}^n \to \mathbb{P}^n \) is closed, \( \pi(\mathcal{Y}) = \{x + \lambda y \mid y \in \mathbb{P}^n, \lambda \in \mathbb{P}^n \} \).

2) WLOG, \( X \) can assume \( \mathbb{P}^m \).

Because closure is a basic property, consider \( X \times Y \) by affiness set.

ETS. Let \( \Delta = \{(x, y) \mid x \in \mathbb{P}^n\} \) is closed in \( \mathbb{P}^n \times \mathbb{P}^n \) since \( X \times \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n \) is closed. Then \( \sigma^{-1}(\Delta) = \mathcal{Y} \).

But \( \Delta = \{(x, x) \mid x \in \mathbb{P}^n\} \) is closed in \( \mathbb{P}^n \times \mathbb{P}^n \)

\[
\begin{bmatrix}
(x_0, x_1, \ldots, x_n) \\
(x_0, x_1, \ldots, x_n)
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_0 \\
y_1 \\
y_2
\end{bmatrix}
\]

\( = \mathbb{P}^n \times \mathbb{P}^n \)

Proof of theorem 2 (Haw/5.2)

1) Relate to the core \( X \times \mathbb{P}^n, Y = \mathbb{A}^m \) (easy) (B. rational).

Point is to show \( \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m \) is a closed map.

\( X \times \mathbb{P}^n \to \mathbb{A}^m \)

\( W = \pi^{-1}(\pi(W)) \)

\( W = \mathbb{P}^n \times \mathbb{A}^m \)

\( \mathbb{P}^n \times \mathbb{A}^m \)

\( q_i \) are homogeneous in \( x_i \).

What \( \lambda \in (\lambda_1, \ldots, \lambda_m) \in \mathbb{A}^m \) are in the image of \( \pi \)?

\( \lambda \in \pi(W) \iff \text{the homogenous poly}(y_1, \ldots, y_m) \ldots \)
have a common non-zero solution.

Look at the set \( T = \{ (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = 0 \} \) and note that the system of equations \((*)\) has a non-zero solution \( \text{iff} \) \( \det(q_i, \ldots, q_n) \neq 0 \). In general, \( (*) \) does not have a solution if \( \det(q_i, \ldots, q_n) = 0 \).

Consider a monomial of degree \( s \):
\[
X_0^{a_0} X_1^{a_1} \cdots X_n^{a_n} = \sum_{i=1}^{n} f_i q_i^{a_i}
\]
for some \( f_i \in k[X_0, \ldots, X_n] \).

Let \( T_f \subseteq T \). Just as in \( \ref{9} \), \( T \times T \) consists of a matrix formed from the columns of \( \phi \) that vanish, there is a \( \sigma \)-dim \( s \) with \( s \geq 1 \) and monomials of degree \( s \).

\[\text{Example} \quad W(y - x^2) = V \subseteq \mathbb{A}^2\]

Preimage of \( \lambda \in R_{x'} \), \( \pi^{-1}(\lambda) = \{ (\lambda, x) \} \) a set of \( 2 \) for \( \lambda \) in \( S \).

Look at corresponding map of coordinate rings:

\[
k[t] \xrightarrow{\pi^*} k[x, y] \quad \xrightarrow{(y = x^2)} \quad k[x, y] \xrightarrow{\phi_{x, y}} k[y, x]
\]

The map is an integral extension of \( \sigma \).

\[t \quad \xrightarrow{\phi} \quad t \phi = y \]

We know \( x, y \) are a \( \lambda \)-basis generator of \( \mathbb{A}^2 \) and \( y = x^2 \).

**Definition:** A map of rings \( A \rightarrow B \) is an integral extension if it is injective and, thinking of \( A \) as a subring of \( B \), every \( \phi(x) \in B \) satisfies some polynomial of the form:

\[
l^n a_b x^n + \cdots + a_0 = 0
\]

for some \( a, b \in A \).

Recall: To check that \( \phi \) is an integral extension, it is enough

\[\Phi \text{ to check that each } A \text{-algebra generator for } B \text{ satisfies a monic polynomial as above.}\]
**Definition** Let \( X \to Y \) be a regular map of affine varieties. We say \( F \) is finite if the corresponding map of coordinate rings \( k[Y] \to k[X] \) is an integral extension.

In particular, if \( F \) is finite, then \( F \) is dominant.

**Theorem** Every finite map is surjective. (proof left as an exercise.)

**Definition** A regular map \( X \to Y \) between affine varieties is said to be finite if for all \( p \in Y \), \( F^{-1}(p) \) is affine and the map \( V_p \to F^{-1}(p) \) is finite as a map of affine varieties.

In Def. 2, it is enough to check for any affine open of \( Y \), say \( U_2 \), that \( V_2 = F^{-1}(U_2) \) is affine and \( V_2 \to F^{-1}(p) \) is finite as a map of affine varieties.

**Theorem** (Finiteness is a local property.) If \( X \to Y \) is a finite map of affine varieties and \( p \in Y \) then every finite map of affine varieties \( F^{-1}(p) \to V_p \)

\( \iff \) if \( V_p \) is any affine open around \( p \).

**Fact** If \( X \to Y \) is a finite map of affine varieties, then \( F^{-1}(p) \) is finite.

**Proof** Let \( X \) and \( Y \) be affine, \( X \subset A^n \), \( Y \subset A^m \) closed sets.

Then \( k[Y] \subset k[X] \) is integral.

\( \iff \) \( F \)

Take \( p = (\alpha_1, \ldots, \alpha_m) \in Y \), say \((\alpha_1, \ldots, \alpha_m) \in F^{-1}(p)\). 
We'll show that there are only finitely many possibilities for each coordinate \( X_i \).

Because \( X_i \subset k[X] \) and \( k[X] \) is integral over \( k[Y] \), \( F \) is a poly.

\( X_i^e + a_1 X_i^{e-1} + \ldots + a_m = 0 \) \( (\text{monic} - X) \)

Evaluate at \( q \cdot F \):

\( X_i^e + a_1 (X_{i-1})^{e-1} + \ldots + a_m (F(q)) = 0 \)

Evaluate at \( q \cdot (F(1)^{e-1} + \ldots + a_m (F(1)^m)) \).

Thus \( \begin{array}{c}
\text{a set of } \frac{\text{polynomials}}{\text{deg} X_i}
\end{array} \)
EXAMPLE  Let \( V \subseteq \mathbb{P}^n \) be a projective variety. Let \( \pi : \mathbb{P}^n \to V \), fix any hyperplane \( H = \mathbb{P}^{n-1} \subseteq \mathbb{P}^n \). Then the projection from \( \pi \) \( \pi \) is a finite map onto its image.

****

NOETHER NORMALIZATION THEOREM

Every irreducible projective variety is a finite cover of some projective space \( \mathbb{P}^d \) (i.e., there is a finite map \( X \to \mathbb{P}^d \)).

Proof: Project from \( p \in \mathbb{P}^n \) to get a finite map \( \mathbb{P}^n \to \mathbb{P}^n \). Let \( \pi(p) \in \mathbb{P}^d \), project again from some \( q \in \mathbb{P}^d \).

10/8/01 \( \pi^{-1}(p) \) is finite since \( \pi \) is finite.

DIMENSION

Basic properties:

1. A point has dimension 0.
2. \( \mathbb{A}^n \) has dimension \( n \), and so does \( \mathbb{P}^n \).
3. If \( X \to Y \) is finite, \( \dim X = \dim Y \).
4. The dimension of a hypersurface \( \mathbb{W}(F) \subseteq \mathbb{A}^n \) (or \( \mathbb{P}^n \)) is \( n-1 \).
(5) If $W$ and $V$ are irreducible and $W \subseteq V$ as an closed subset of $V$, then $\dim W < \dim V$.

(6) $\dim (X \cdot Y) = \dim X + \dim Y$.

(7) If $V$ is irreducible, and $U \subseteq V$ is a nonempty open subset, then $\dim U = \dim V$.

(8) For $V$ irreducible, $\dim V = \max \dim$ of the irreducible components of $V$.

For $V$ to be an irreducible affine variety:

Recall: The rational function field of $V$ denoted $k(V)$ is the field of all rational functions. Equivalently, $k(V)$ is the field of meromorphic functions on a nonempty, open, affine subset of $V$.

Let $U$ be a nonempty, open, affine subset of $V$.

REMARK. The dimension of $V$ is the transcendence degree of $k(V)$ over $k$.

REMARK. If $k \subseteq k'$ is any field extension, then let $x \mapsto \sigma(x)$

\[ k[V] = k(V) \cong k[U] = k[U_1] \cong k[U_1 \cdots U_n] \cong k[A^n] \]

For $Y \subseteq X$ and $x \in X$, let $x \in X$ be a nonempty, open, affine subset.

Thus $k[V] \subseteq k[X]$. The natural extension.

$\Rightarrow$

$\dim k(V)$ is irreducible.

For (4), reduce to $W(F) \subseteq A^n$. Factor $F$, uniquely into irreducible polynomials.

$F = F_1 \cdots F_n \quad W(F) = W(F_1) \cup \cdots \cup W(F_n)$

Each $W(F_i)$ is irreducible.

Therefore, $V \cap W(F) \subseteq A^n$. Say $X \subseteq V$ is an irreducible affine variety.

Then $\dim X = \dim X_1 \cup \cdots \cup X_n$.

(1) $X_{i_0} \subseteq X_{i_1} \cup X_{i_2}$. Let $X_{i_1}$ be an irreducible affine variety.

If $X_{i_1}$, then $\exists x_i \in X_{i_1}$. If not, then $X_{i_1}$ is not irreducible.
Recall: For an irreducible variety, \( \dim V = \text{transcendence degree } k(V) \)

\[
\begin{align*}
\text{If } V & \rightarrow P^n \\
\text{Then } & \dim k(V) = \dim k(x_0, \ldots, x_n) \text{ algebraic.} \\
\text{Note that } m & \text{ is the transcendence degree of } k(V) \text{ over } k \Rightarrow m = \dim V.
\end{align*}
\]

**FACT** If \( V \) is an irreducible variety over \( k \), then \( V \) has a nonempty open set which is "smooth," so can be given a structure of a complex manifold. Then \( \dim V \) is the complex dimension of this manifold.

**THEOREM** \( X \rightarrow Y \), then \( \dim X \leq \dim Y \), with equality if \( Y \) is irreducible, then \( \dim X = \dim Y \) if \( X = Y \).

[In other words, \( \dim X = \dim Y \Rightarrow X \) is a component of \( Y \) of maximal dimension]

**Proof:** WLOG, can assume \( X \) and \( Y \) are affine, irreducible.

\[
X \rightarrow Y \subseteq \mathbb{A}^n \text{ is closed and finite.}
\]

\[
k(X) \subseteq k[Y] \text{ (by exercise,)}
\]

Fix a transcendence basis \( x_1, \ldots, x_\ell \) in \( k[X]/k \) (may assume \( x_1, \ldots, x_\ell \) are coordinates for \( \mathbb{A}^\ell \)). They must be alg. independent considered as elts of \( k(Y) \Rightarrow \dim Y \geq \dim X \).

To show \( \dim X = \dim Y \Rightarrow X = Y \), ETS the injection \( k[Y] \rightarrow k[X] \).

Say \( u \in k[Y] \) s.t. \( u \notin \text{ker} \). Note: \( x_1, \ldots, x_\ell \) are a transcendence basis for \( k(Y)/k \) \( \Rightarrow u \) must be alg. dependent on them \( \Rightarrow u \in k[X] \) with coefficients.

\[
\begin{align*}
G &= a_\ell(x_1, \ldots, x_\ell) u^\ell + a_{\ell-1}(x_1, \ldots, x_\ell) u^{\ell-1} + \ldots + a_0(x_1, \ldots, x_\ell) \in k[X] \text{ algebraically.} \\
\text{(clearly denominators can assume } a_0 \text{ are poly's)} \Rightarrow \text{ assume } a_\ell \neq 0. \\
\text{Further to } X \text{, note } u(x) = 0 \text{ as well.}
\end{align*}
\]

\[
\Rightarrow a_\ell(x_1, \ldots, x_\ell) = 0 \text{ contradict, \( \Rightarrow X = Y \).}
\]
ALTERNATIVE DEFINITION OF DIMENSION

Let \( V \) be an arbitrary variety. Then \( \dim V \) is the length of the longest possible chain of closed irreducible subvarieties:

\[
\text{dim} V = V_0 \supset V_1 \supset \cdots \supset V_n
\]

By the theorem we just proved, the dimension of each \( V_i \) is strictly greater than that of its predecessor \( V_{i-1} \).

Note: \( \text{dim} V = n \).

Example:

\[
A^n \supset A^r \supset A^2 \supset \cdots \supset A^1
\]

REMARK: If \( V \) is affine, then each \( V_i \) corresponds to a prime ideal in \( k(V) \) and \( \dim V = \text{rank} \) of the dimension of \( k(V) \).

Full proof that \( \text{dim} V \) is the length of the longest chain of irreducibles, by induction on \( V \).

We'll do it when \( V \) is projective (good for intuition, irreducible).

Step 1: For any finite set of points in \( P^n \), there is a hyperplane that misses them all. (Equivalently, there is a linear form \( l \) which doesn't vanish at any of them.)

Step 2: Fix \( V \) (not necessary irreducible) \( \subseteq P^n \), project \( V \) a linear form \( l \), which doesn't vanish on any component of \( V \) (pick a point \( p \) on each component \( V_i \), take \( l \), as in step 1).

\[
V = V_0 \supset V_1 \supset \cdots \supset V_n \supset V_{n+1} = \emptyset
\]

Now, repeat, get \( V_2 = V_1 \cap W(l_2) \), get a chain \( V_2 \supset V_3 \supset \cdots \).
Look at $l_1, \ldots, l_n$. There are linear forms where common zero set is $0 \in V$. This defines a map

$$V \to \mathbb{P}^d$$

$$x \mapsto [l_1(x), \ldots, l_n(x)]$$

This is a finite map onto its image, $\text{dim} V \leq d$.

Say the image each answer from $d$?

Then get a map $V \to \mathbb{P}^d$, it's finite and surjective $\Rightarrow \text{dim} V \leq d$.

This is the contradiction.

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Reading: Catch-up. All of Schubert Part I (§1.2.3.7.10.6)

19/12/01

Intuition:

1) $V \subset \mathbb{P}^n$ projective of dim $n$, $V \cap H$ where $H$ is a "generic hyperplane" has dim $n-1$.

[Consider the space of all hyperplanes in $\mathbb{P}^n$, $\mathbb{P}^n$

$$L = \{v(ax_0 + \ldots + ax_n) \mapsto [x_0 : \ldots : x_n]$$

Fix any $p \in \mathbb{P}^n$, there is a hyperplane passing through $p$ from a subset of $\mathbb{P}^n$.

$$\mathbb{P}^n \subset \mathbb{P}^n$$

2) $V \subset \mathbb{P}^n$ projective of dim $n$. Then $V \cap W(F)$ has dim $n-1$, where $F$ is a linear space of dimension $m$, $m$ degree of $V$.

3) Say $V \subset \mathbb{P}^n$ is defined by $r$ homogenous polynomials.

$$\text{dim} V \geq n-r$$

4) An irreducible variety in $\mathbb{P}^n$ of dim $n-1$ is always a hypersurface i.e. $V = W(F)$. 

5) $\text{dim} V = N-r$, so say $V$ is a complete intersection.
Theorem

Theorem on Dimension of Fibers of Regular Map

Example

\[ \mathbb{A}^N \rightarrow \mathbb{A}^m \]

\[ (x_1, \ldots, x_N) \rightarrow (x_1, \ldots, x_m) \]

Fiber over a point \( p \in \mathbb{A}^m \):

\[ \pi^{-1}(p) = \{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_N) | x_i = 0\} = \mathbb{A}^{m-n} \]

Dimension of fiber:

\[ \dim \text{fibre} = \dim \text{source} - \dim \text{target} \]

Example

\[ \mathbb{A}^3 \rightarrow \mathbb{A}^2 \]

\[ (x_1, x_2, x_3) \rightarrow (x_1, x_2) \]

Fiber over typical point \( (a, b) \in \mathbb{A}^2 \):

\[ \pi^{-1}(a, b) = \{(x_1, x_2) | x_3 = 0\} = \mathbb{A}^2 \]

Dimension:

\[ \dim \text{fibre} = \dim \text{source} - \dim \text{target} = 3 - 2 = 1 \]

Theorem

Let \( X \rightarrow Y \) be a surjective regular map of projective varieties.

Then

1) \( \dim X > \dim Y \)

2) \( Y \) is a nonempty open set \( U \).

3) For any fiber \( \pi^{-1}(y) \) has dimension \( \dim X - \dim Y \).

4) The set \( Y \) consists of \( y \) such that \( \dim \pi^{-1}(y) = \dim X - \dim Y \).

Corollary

Let \( X \rightarrow Y \) be a surjective regular map of projective varieties.

Then if \( Y \) is irreducible and \( \forall y \in Y \) fibers are isomorphic of the same dimension (at each \( y \)), then also \( X \) is irreducible.
General Question: Let $X$ be a surface in $P^3$ defined by $F_0$ of
degree $m$. How many lines lie on $X$? Are there any at all?

$m=1$
Lots

$m=2$
Typical line $\equiv W(x^2-y^4) \subseteq P^3$

$m=3$ Always at least one line. Typical surface has 

$m=4$ Typical surface has no lines.

Look at parameter space of all surfaces of degree $m \in P^3 \rightarrow P^{(m^{2}+3)}/1$.

Look at set $G$ of all lines in $P^3$ (called the Grassmannian $G_2(k^N)$)

$G$ is an irreducible projective variety of dim 4.

Look at $Z_m \subseteq G \times P^{(m^{2}+3)/1}$

$\{(l, x) | l \in X\}$ This is a closed subvariety of $G \times P^{(m^{2}+3)/1}$.

There are 2 projections:

$Z_m \rightarrow \Pi_1 \rightarrow \Pi_2 \rightarrow P^{(m^{2}+3)/1}$

Take a surface in $P^3$ (i.e., a point in $P^{(m^{2}+3)/1}$). Then $X$ contains a
line iff $\Pi_2^{-1}(x) \neq \emptyset$.

Fix $l \subseteq G$. What is $\Pi_2^{-1}(l)$? WLOG $l = V(x_0, x_1)$. A surface contains

$l$ iff $(x_0, x_1) \in \mathbb{F}$ iff $F$ has no monomial terms involving

$x_0, x_1, x_2, x_3, \cdots, x_{m}$

in $F$.

The set of surfaces $X$ which contain $l$

form a closed linear subspace subvariety

in $P^{(m^{2}+3)/1}$ of codimension $m+1$.

\[ \dim (\text{Fiber of } \Pi_1) = (m^{2}+3)/1 - m + 1 \]  
\[ \dim Z_m = (m^{2}+3)/1 - (m+1) + 4 \]
Tangent Space on affine variety $V = V(I(V)) \subseteq \mathbb{A}^n$ at $0 \in \mathbb{A}^n$

Idea: Tangent space will be the union of all pts $l$ all line sections $p(a, l)$ at $0$.

Example: $V = V(y-x^2) \subseteq \mathbb{A}^2$

Intuition: $l$ intersects $V$ with multiplicity $> 1$ at $(0, 0)$.

Formally: Parameterize $l = \{(ta, kb) | t \in k\}$.

Compute $V \cap l$. Solve $f - (ta)^2 = 0$, where $f(0, 0) = 0$.

There's a "multiple" intersection pt iff $t^2$ divides the system.

[Here, the only tangent line is $y = 0$, i.e., $l = \{(0, a) | a \in k\} = V(y)$.]

Let $V \subseteq \mathbb{A}^n$ be an arbitrary closed set.

Take any $l(a, b) = \{(ta, kb) | t \in k\}$.

What does $l$ intersect $V$ with multiplicity $\geq 2$ at $0$?

Compute $V \cap l$.

So, solve for $t$ with the highest common factor of $f$, i.e., $\gcd(1, (t-a)^2) = 1$ in $k[[t]]$.

The intersection pts of $V \cap l$ are given by $t_a$, and the multiplicity at the pt. corresponding to $t_a = a$. 
**Definition** The intersection multiplicity of $L$ with $V$ at the origin is the highest power of $t$ dividing all $g_i(x,t)$ in the system $L_i$.

**Definition** A line $L$ is tangent to $V$ at origin if its intersection multiplicity with $V$ at the origin is 2.

**Definition** The tangent space $T_p V$ at $p$ is the vector space of all points lying on lines tangent to $V$ at $p$.

**Caution** $V(y^2 - x^3)$ is not the tangent space.

**Fact** The tangent space to $V$ at $p$ is a linear variety in $A^n$ passing through $p$.

Intuitively, this is really the tangent space if $V$ is smooth at $p$.

**Definition** Let $p \in V$ be a point of intersection. Define $V_p = \text{ideal of equations containing } p$. Then $p$ is a smooth point of $V$ if $\text{dim } V_p = 0$.

Prove that $T_p V$ is a linear variety in $A^n$.

$(y_1, y_2) = T_p V \bigoplus V$. Take each $g_i = L_i + H_i$.

If $\text{dim } V > 2$ then $L_i + H_i$.

$g_i(a,t) = L_i(a) + H_i(a,t) = L_i(a) + t^2 H_i(a,t)$.

The line $l = \text{all } t\text{ such that } L_i(a) + t^2 H_i(a,t) = 0$.

This shows $T_p V = V(l_1, ..., l_n)$ - linear variety.

To write down the equation for $T_p V$ at existing $p \in V$, expand each $L_i(x_1, ..., x_n) + H_i(x_1, ..., x_n)$. 

$(x_1, ..., x_n)$.
\[ \frac{\partial y_i}{\partial x_p} (x - \lambda_1, \ldots, x_n - \lambda_n) \]

\[ \frac{\partial y_i}{\partial x_p} (x - \lambda_1) + \frac{\partial y_i}{\partial x_p} (x_2 - \lambda_2) + \ldots + \frac{\partial y_i}{\partial x_p} (x_n - \lambda_n) \]

\[ p \in T_p V = W \left( \frac{\partial y_1}{\partial x_1}, \ldots, \frac{\partial y_n}{\partial x_n} \right) \leq M^n \]

**Non-obvious fact:** \( \dim(T_p V) \geq \dim V = \text{dimension of the largest linear component of } V \text{ that contains } p \) (proof omitted)

**Easy check:** that \( T_p V \) is independent of the choice of generator in \( T(V) \).

**Proof:** \( (y_1, \ldots, y_n) = (h_1, \ldots, h_n) \) where \( y \in R, h \in R^n \) and \( p = x \in R^n \)

\[ h = s, g_1, \ldots, g_n x \in \mathbb{R}^n \]

\[ \frac{\partial y_i}{\partial x_p} = (d_1, \ldots, d_n) \text{ where } d_i = \frac{\partial y_i}{\partial x_p} \text{ for } i = 1, \ldots, n \]

**Example:** \( V = W(y^2 - x^2, z^2) \)

\[ T_p V = W(0) = M^0 \quad \text{(no linear term)} \]

**Example:** \( V = W(x^2 + y^2 + z^2 - 1) \leq M^3 \)

\[ p = (0, 0, 1) \]

\[ w(2) \]

\[ T_p V = W(1, 1, 1, 2, 2, 2, 1) \leq M^3 \]

\[ \frac{\partial y_i}{\partial x_p} = \frac{2 \xi}{\partial x_1} p(x - \lambda_1) + \frac{\partial y_i}{\partial x_p} (x_n - \lambda_n) \]

\[ = 2z - 2 \]

10/13/01

Not for lecturing. Just as an affine variety is essentially the same thing as its coordinate ring, then a vector bundle object, called the local ring of \( V \) at a pt. \( P \), which records all the geometric information at \( V \) on a model of \( P \).
**Definition**  For an irreducible variety $V$, and a point $p \in V$, the local ring of $V$ at $p$ is the subring of $k(V)$ which are regular at $p$. Notation: $O_{V,p}$.

$$O_{V,p} = \left\{ \frac{f}{g} \in k(V) \mid g(p) \neq 0 \right\}$$

**Example**  $V = \mathbb{A}^2$, $p = (0,0)$, $k(V) = k(x,y)$, function field of $k[x,y]$.

$O_{V,p} = \left\{ \frac{f}{g} \mid f, g \in k[x,y] \text{ with } g(0,0) \neq 0 \right\}$, i.e., ring of regular functions at $p$.

Include $\frac{x}{y}$, does not include $\frac{y}{x}$.

$O_{V,p} = \lim_{U \to p} O_V(U)$

General commutative algebra:

$A_{m,g}$

$U_{m,g}$

$P_{m,g}$

**Definition**  The local ring of $A$ at $P$, denoted $A_P$, is the set of fractions

$$\left\{ \frac{f}{g} \mid f, g \in A \text{ and } g \neq 0 \text{ at } P \right\}$$

$$\frac{f}{g} = \frac{b}{k} \iff \exists u \in P \text{ s.t. } u(fk-gh) = 0 \iff \frac{f}{g} = \frac{u^* f}{u^* g} = \frac{b}{k}$$

**Note:** $A_{m,g}$ forms a ring with "the usual operations" of addition and multiplication.

For $p \in V$, define $O_{V,p}$ as follows: take any open affine $U \subseteq V$ containing $p$.

$$O_{V,p} = (k[U])_{\mathfrak{m}_p}$$

where $\mathfrak{m}_p = \mathfrak{m}(p)$, i.e., maximal prime ideal containing $p$.

$O_{V,p}$ is independent of the choice of affine cover.
Basic properties

1. \( A_p \) is a local ring. Let it have a unique maximal ideal \( \mathfrak{m} \).

(Thus, a maximal ideal is \( \frac{1}{2} \) for \( f \in \mathfrak{m}, \mathfrak{m} \notin \) any element in \( \mathfrak{m} \) for a unit)

2. There is a natural map \( A \to A_p, f \to \frac{f}{1} \).

3. If \( A \) is Noetherian, then \( A_p \) is Noetherian (that they are not \( f, g \in \mathfrak{m} \)).

\( V = V(x^2 + y^2 - z^2) \subseteq A^3 \)

\[ \mathcal{O}_{A_p} = \frac{k[x, y, z]}{(x^2 + y^2 - z^2)} \]

\[ \mathcal{M}_{A_p} = \text{Maximal ideal} = \left\{ \frac{1}{2}, [x, y, z] \right\} \]

II Zariski Tangent Space

\( V \subseteq A^N, \quad \mathbb{I}(V) = (g_1, \ldots, g_r) \)

Recall: \( T_p V = V(\text{d}g_1, \ldots, \text{d}g_r) \subseteq A^N \) (Think of \( p \) as an element in \( A_p \) with \( g_r \) in the field of fractions by \( p \))

Note: there is a natural map given by \( \text{d}p: k[V] \to (T_p V)^* \) (duale Elements)

\[ \text{d}p \mapsto \text{d}f \mid T_p V \]

(Recall \( \text{d}f = \frac{\partial f}{\partial x, y, z} \))

Basic properties of \( \text{d}p \)

\[ \text{d}(fg) = \text{d}f \cdot g + f \cdot \text{d}g \]

\[ \text{d}(f(g)) = \text{d}(f)(g) \cdot \text{d}g \cdot (\text{d}(f)g(p) + \frac{\partial f}{\partial x, y, z}) \cdot \text{d}g. \]

Note. \( \text{d}p \) is surjective. Given a linear function \( \lambda \) on \( (T_p V)^* \) from \( \mathbb{A}^N \)

\[ (T_p V \subseteq \mathbb{A}^N) \Rightarrow (\mathbb{A}^N \to (T_p V)^* \text{ surjective}) \]

\( q = x(x-1)y = x^2(x-1) \) is \( \text{d}p \) - surjective. In fact \( \text{d}g = \text{d}p = \text{d}g \cdot \text{d}p \)

So \( \text{d}p \) restricted to a surjection \( \mathbb{A}^N \to (T_p V)^* \)

\[ q \to \text{d}g \mid T_p V \]
**Theorem:** \( \varphi \) induces an isomorphism \( \frac{M_p}{M_p^2} \cong (T_p V)^* \) of vector spaces.

**Proof:** Only need to check homomorphism \( \varphi \). 

**Definition:** The Zariski tangent space to \( V \) at \( p \) is the \( k \)-vector space \( (\frac{M_p}{M_p^2})^* \), where \( M \) is the unique maximal ideal of the local ring \( \mathcal{O}_{V,p} \).

**Fact:** If \( V \xrightarrow{f} W \) is a regular map of \( k \)-varieties, then there is a unique map \( \varphi \) on the duals of the fibered tangent spaces:

\[
T_pV \xrightarrow{\varphi} T_{f(p)}W
\]

(\( k \)-vector spaces).

To check homomorphism properties, choose \( \varphi \) so that:

\[
\begin{align*}
\varphi & : \frac{M_p}{M_p^2} \to \frac{M_{f(p)}}{M_{f(p)}^2} \\
\varphi & : \frac{m_p}{m_p^2} \to \frac{m_{f(p)}}{m_{f(p)}^2}
\end{align*}
\]

Define it:

\[
(\frac{M_p}{M_p^2})^* \to (\frac{m_{f(p)}}{m_{f(p)}^2})^*
\]

10/19/01 Smooth & singular points, singular locus of a variety

\( V \) - any \( k \)-variety.

Recall: A pt \( p \in V \) is smooth if \( \dim T_p V = \dim_p V \). Otherwise, \( p \) is a singular point of \( V \).

Recall: Think of \( T_p V \) as in affine case: \( T_p V = \mathcal{O}_{V,p} / (\partial_{p_1}, \ldots, \partial_{p_n}) \subset \mathbb{A}_k^n \), where \( V \subset \mathbb{A}_k^n \), closed set \( I(V) = \langle g_1, \ldots, g_r \rangle \).

**Theorem:** The locus of singular points of a variety \( V \) is a proper closed subset of \( V \). Explicitly, if \( V \subset \mathbb{A}_k^n \) is an irreducible closed set & \( I(V) = \langle g_1, \ldots, g_r \rangle \) then...
Sing (V) = W (\text{cones} \text{ of} \text{ tangent} \text{ space}) \cap V \subseteq A^r,

where \( c \) = \text{codim of} \ V \text{ in} \ A^N.

Also, if \( V \) is projective & irreducible \( V \subseteq P^N \), \( \Pi (V) = (s_0, \ldots, s_r) \) homogeneous ideal, then \( \text{Sing} \ V = V (\text{cone} \text{ of} \text{ minors of} \ (s_i) \cap V \subseteq P^N \).

\[ \text{THEOREM 2} \]

The dimension of \( T_p \ V \) at a singular point \( p \) of \( V \) is strictly greater than \( \dim_p V \).

\[ \text{EXAMPLE} \]

\( V (x^2 + y^2 - z^2) \subseteq A^3 \)

\( T_p V = W (d_p (x^2 + y^2 - z^2)) \subseteq A^5 \)

\[ W (2a(x-a) + 2b(y-b) + 2c(z-c)) = W (a(x-x) + b(y-y) + c(z-z)) \]

\( \dim T_p V = 2 \) for \( p \neq (a, b, c) \), in which case \( \dim T_p V = 3 \).

Sing (V) = W (1 \times 1 \text{ minors of} \ [x^2, y^2, z^2]) \cap V = V (x^2, y^2) \cap V = (v_{x,v})

\( W (x^2 + y^2 - z^2) \subseteq P^2 \)

\( \text{Sing} \ (V) = W (d_{x,y} (x^2 + y^2 - z^2)) = W (x^2, y^2) \cap V = \emptyset \)

\[ \text{PROOF} \]

Assume \( V \) is affine (closed in \( A^N \) but \( \dim V < N \)).

Consider the \( \bar{X} = \{ (p, \bar{y}) \mid p \in T_p V \} \subseteq V \times A^N \). Easy to check that \( \bar{X} \)

\( \text{is a closed subspace of} \ V \times A^N \)

\( \bar{X} \mapsto (p, \bar{y}) \quad \text{for} \quad p \in V \}

\[ T_p V = T_p \bar{X} \]

By the known on the dimension of fibers, the subset \( S \subseteq V \) of points whose fibers have minimum positive dimension is open. We want to show this minimum positive dimension = \( \dim_p V = \dim_p V \) (on \( V \) itself).

\[ \text{Use FACT (Ennoin) Every irreducible variety contains a non-empty open subset which is isomorphic to a non-closed open subset of a hypersurface.} \]

So, we can assume \( V \) is hypersurface in \( A^{N-1} \) (or \( V = \emptyset \)).
Let $V = V(g_1, \ldots, g_r) \subseteq \mathbb{A}^N$.

**Proof:**

- **Step 1:** Since $V$ is affine, irreducible, $V = V(g_1, \ldots, g_r) \subseteq \mathbb{A}^N$.

**ETS:** Suppose $V = V( \text{cyclic minors of } \text{Jac})$. Then $V = V : \dim(\mathbb{V}(\mathbb{C})) > d \implies \dim(\mathbb{V}) > d = \dim(\mathbb{V})$.

- **Step 2:** Consider $V(d_1g_1, \ldots, d_rg_r)$.

- **Step 3:** Let $p = (x_0, \ldots, x_N)$.

- **Step 4:** Define $d_i g_i = \frac{\partial}{\partial x_i} (x_i - \lambda_1, \ldots, x_i - \lambda_r)$.

- **Step 5:** The solutions to the linear system of equations $T_p V$, the solutions to the linear system of equations $\left[ \begin{array}{c} d_1 g_1 \\ \cdot \\ \cdot \\ d_r g_r \end{array} \right] \cdot \left[ \begin{array}{c} \frac{\partial x_0}{\partial x_1} \\ \frac{\partial x_0}{\partial x_2} \\ \vdots \\ \frac{\partial x_0}{\partial x_N} \\ \frac{\partial x_1}{\partial x_0} \\ \vdots \\ \frac{\partial x_N}{\partial x_0} \end{array} \right] \left[ \begin{array}{c} x_0 - \lambda_1 \\ \vdots \\ x_N - \lambda_r \end{array} \right] = 0$.

- **Step 6:** $\dim(\ker(f)) = \dim(T_p V) = N - \text{rank } (J_{\text{Jac}})$.

- **Step 7:** $\text{rank}(J_{\text{Jac}}) < N - d = \dim(V) \implies \text{All cyclic minors vanish.}$

**10/23/01 Inv. 6**

**Projective Tangent Spaces:**

Let $V \subseteq \mathbb{P}^N$ be a projective variety. $I(V) = (g_1, \ldots, g_r) \subseteq k[x_0, \ldots, x_N]$, fix $p = [a_0 : \ldots : a_N]$.

Several ways to think about projective tangent spaces:

- **Projective tangent space $T_p V$** (continue same notation as affine tangent space).

1. The set of all points $q \in \mathbb{P}^N$ lying on (projective) lines on $\mathbb{P}^N$ that are tangent to $V$ at $p$.

   (Recall: A line $\ell \subseteq \mathbb{P}^N$ is tangent to $V$ at $p$ if $\mathfrak{L}_V|_{\ell}$ has intersection multiplicity $\geq 2$, think of restricting the poly to $q$, then each will factor completely into linear polynomials, and one factor corresponds to point $p$. Thus factor must appear with an exponent $\geq 2$).

2. The scheme in $\mathbb{P}^N$ of the tangent cone to $V$ at $p$.

   The scheme of $\mathbb{P}^N$.
3) Let $\tilde{V} \subseteq \mathbb{A}^{n+1}$ be the affine cone over $V \subseteq \mathbb{P}^n$ ($\tilde{V} = W(q_0, \ldots, q_r) \subseteq \mathbb{A}^{n+1}$)

Let $p$ be any representative point representing $p$, i.e., $\tilde{p} = (1, 0, \ldots, 0)$

Let $T_{\tilde{p}}\tilde{V} \subseteq \mathbb{A}^{n+1}$ be the tangent space to $\tilde{V}$ at $\tilde{p}$. This is a vector space of $\mathbb{k}^{n+1}$ (i.e., it passes through the origin). The projective tangent plane to $V$ at $p = \mathbb{P}(T_{\tilde{p}}\tilde{V}) \subseteq \mathbb{P}(\mathbb{k}^{n+1}) = \mathbb{P}^n$

4) Look at the Jacobian matrix $\left( \frac{\partial q_i}{\partial x_j} \right)_p$

\[ \ker (J_{\mathbb{P}}(p)) \subseteq \mathbb{k}^{n+1} \]

(We defined up to order at last well-def)

Projective tangent space $= \mathbb{P}(\ker (J_{\mathbb{P}}))$

Rewrite

\[ W(x^2 + y^2) \subseteq \mathbb{A}^3 \]

$\mathbb{P}(T_{\tilde{p}}\tilde{V}) = W(ax^2 + by^2) \subseteq \mathbb{P}^2$

If $t \neq 0$ in $\mathbb{P}^2$, put $x, y$

[Causes map]

$V \subseteq \mathbb{P}^n$, smooth projective variety, dim $d$ (irreducible)

$V \overset{\pi}{\rightarrow} \text{Gr}(d, \mathbb{A}^{n+1})$ (Grassmannian of $d$-planes in $\mathbb{A}^{n+1}$)

$\pi : (V, \tilde{p}) \rightarrow \text{proj} T_{\tilde{p}}V$. This is regular

For $C \subseteq \mathbb{P}^2$, a smooth plane curve $C \rightarrow \mathbb{P}^2$, degree $d$ in $\mathbb{P}^2$.

$p \rightarrow \text{proj} T_p C$
\[ C = W(x^2 + y^2 - z^2) \]

If \( V \subseteq \mathbb{P}^N \) is not a linear subspace of \( \mathbb{P}^N \), then the maps

\[ \varphi : \mathbb{P}^N \to \mathbb{P}^N \]

are a finite (in particular it image a variety of the same

dimension)

For \( d \geq 2 \), the image \( \mathcal{C} \) is called the dual curve

\[ \varphi(p) = \mathcal{C}(ax + by + cz) = [a : b : c] \in \mathbb{P}^2 \]

For \( d \geq 3 \), not a monomial, not even 1-4 in grade, dual curve is not smooth

\[ \varphi(p) = \mathcal{C}(p) \]

10/24/01 Families of Varieties

Example: Fix \( a,b,c \), look at \( V_{a} = W(xy - z) \subseteq \mathbb{A}^2 \). Varying \( x \) not yet a "true" parameter family of hyperbolas.

Example: \( V \subseteq \mathbb{P}^N \)

Projective variety.

For each hyperplane \( H \subseteq \mathbb{P}^N \) (\( H \neq H^0 \)). The collection of \( V_H = V \cap H \subseteq H^0 \)

the family of hyperplane sections of \( V \).

Definition: A family is a surjective map of space varieties \( X \xrightarrow{\pi} B \)

The member of the family are the fibers

\[ X_p = \pi^{-1}(p) \quad \text{dies } p \in B \]

The base (or parametering space) is \( B \)

\[ X = W(xy - z) \subseteq \mathbb{A}^3 \]

Fiber over \( A \)

\[ \pi^{-1}(A) = \{ (x,y,x | xy - z) \subseteq \mathbb{A}^3 \]
Example. \( V \subseteq P^n \) projective.

\[
V_H = V \cap H \subseteq \mathbb{P}^{n-1}
\]

\[
X = \{(p, H) \mid p \in H\} \subseteq \mathbb{P}^{n-2}
\]

\[
\pi^*(H) = \{(p, H) \mid p \in H\} \subseteq \mathbb{P}^{n-2} \times \mathbb{P}^n
\]

\[
\{(p, H) \mid p \in H \} \subseteq \mathbb{P}^{n-2} \times \mathbb{P}^n
\]

\[
\lambda \in H \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^n
\]

\[
X = \mathbb{P}(\mathcal{O}(x_1 + \cdots + x_n)) \cap (\mathbb{P}^{n-2} \times \mathbb{P}^n)
\]

**General Principle**. "Nice properties" of varieties are open in families.

In particular, if a variety is smooth, then almost all members are smooth.

**Example**: \( V(xy - z) \subseteq \mathbb{A}^3 \) hyperbolic family

\[
\begin{align*}
\Lambda^2 & = V(xy - z) \subseteq \mathbb{A}^3 \\
\mathbb{A}^1 & = V(x_1) \cap V(x_2) = \emptyset \quad \text{unless } \lambda = 0
\end{align*}
\]

Smooth locus of the family is parameterized by \( \mathbb{A}^1 \) - open.

**Bertini's Theorem**: \( V \subseteq P^n \) smooth, irreducible, \( \text{dim} \geq 2 \).

Then a generic hyperplane section of \( V \) is smooth.

\[
X \subseteq V \times P^n
\]

\[
\{(p, H) \mid p \in H\}
\]

\[
\{(H, V \cap H = X_H) \mid \text{open} \quad \text{(genericity of the lemma)}
\]
If we look at curves in $\mathbb{P}^2$ of degree $d$

\[ C \in \mathbb{P}^2, F = \sum a_i x^i y^j z^k \quad \Rightarrow [a_0, a_1, \ldots, a_d] \in \mathbb{P}^{Nd} \]

Look at the subset $S_d \subset \mathbb{P}^{Nd}$ so that all the curves corresponding to the points of $S_d$ are A-gonal.

Q: Is this a 3-fundamental? Is it open or closed in $\mathbb{P}^{Nd}$?

What is its dimension? What does the set look like anyway?

Auxiliary construction $P \in \mathbb{P}^2 \times \mathbb{P}^{Nd}$

Consider $(p, C)$ where $p$ is a singular point of $C$. 

Relation to $S_d$: $\pi_d(\Gamma_d) = S_d$. Look at "universal equation"

\[ F = \sum a_{i,j} x^i y^j z^k \]

\[ F_x(p) = 0 \quad \sum i a_{i,j} x^{i-1} y^j z^k = 0 \]

\[ F_y(p) = 0 \quad \sum j a_{i,j} y^{i-1} x^i z^k = 0 \]

\[ F_z(p) = 0 \quad \sum k a_{i,j} z^{i-1} x^i y^j = 0 \]

\[ \text{Fiber: } x F_x + y F_y + z F_z = \mathsf{h} \cdot \mathsf{F} \]

\[ \Rightarrow \dim P \geq Nd + 2 - 3 = Nd - 1 \]

Since $P_0$ is a subvariety and $\pi_0(P_0) = S_d$

So $S_d$ is a closed subset of $\mathbb{P}^{Nd}$.

To figure out more about $P_d$, look at the first projection.

\[ \mathbb{P}^2 \quad \text{fiber a?} \]

Plug in $p(0, 0)$ into equation $a_{00} = 0, a_{01} = 0, a_{02} = 0$

So each fiber is $\mathbb{P}^{Nd - 3}$

Conclusion: $P_d$ is of dimension $Nd - 3 + 2 = Nd - 1$. 

$P_d$ is irreducible [indeed, $F_x, F_y, F_z$ are all linear] all plane.
To see what $S_d$ looks like, let's analyze the map $\Gamma_d \xrightarrow{\pi_d} S_d$.
Let's look in local coordinates on $\mathbb{P}^2$ (pick $\mathbb{Z} \neq 0$, and $x = \frac{x}{1}$, $y = \frac{y}{1}$).
In this chart, local equations are: $f(p) = 0$, $\sum a_i \frac{\partial f}{\partial x_i} = 0$.

Tangent space $\Gamma_p = \sum a_i \frac{\partial f}{\partial x_i} = 0$.

Want to look at tangent space to $\Gamma$ at each point.
If you have a variety $V \subset \mathbb{A}^n$, given by ideal $I = (x_1, \ldots, x_n)$, then the tangent space at each point is the kernel of the Jacobian matrix $J_f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$. 

Let's pick $p = (0,0)$ and look at matrix:

$$
\begin{bmatrix}
0 & a_{10} & a_{01} & 1 & 0 & 0 & 0 \\
0 & 2a_{20} & a_{21} & 0 & 1 & 0 & 0 \\
0 & a_{30} & 2a_{31} & 0 & 0 & 1 & 0
\end{bmatrix}
$$

If we have a direction $g = (x', y')$, we see $g = (2a_{20}x', a_{21}x' + 2a_{20}y')$.

Case 1: Matrix is non-degenerate; in this case, by linear change of variables, we can put it in the form $[a_{ij}] \Rightarrow f = x'y' + \text{higher order}$.
If there is a node at $p$, at the level of point, the tangent space is given by curves $y = bx + c$ with $b > 0$, and all other $b$'s arbitrary, and $(x,y)$ uniquely chosen, satisfying the equation:

$$
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
0 \\
a
\end{bmatrix}

If this kind of point $p$ is a local isomorphism (isomorphic in tangent space).

General question: if you have a space parameterizing objects, the local geometry at a point $C$ of the space is determined by the local geometry of the object it represents.

10/29/01: Hironaka's Desingularization Theorem

**Result.** The singular set of a variety is a proper closed subvariety.

**THEOREM** (Hironaka 1964). Assume ground field $k$ has char 0. For any variety $V$, there exists a smooth variety $X$ mapping properly and birationally onto $V$. More precisely, $X \subseteq V \times \mathbb{P}^n$

\[ \begin{array}{c}
\pi \\
V \times \mathbb{P}^n \end{array} \]

$V = \mathbb{P}^n$.

$X$ is a smooth, closed subvariety of $V \times \mathbb{P}^n$ and the projection onto $V$ defines an isomorphism away from the regular set of $V$.

![Diagram of a variety and its resolution](image)

**Remark:** $X$ is not unique.

**Remark:** $X$ is called a resolution of singularities.

Then it's very hard to say what $X$ is explicitly.

**Remark:** Proper base change of varieties are defined over $\mathbb{A}_C^n$, and the preimage of compact sets in $C - \{p\}$ are compact. An example is a curve with a maximal chain.
2) A morphism \( X \rightarrow V \) is **birational** if it restricts to an isomorphism on a dense open subset \( U \) of \( X \), i.e.,

\[
U \cong T(U) \subseteq V
\]

3) If \( V \) is projective, so is \( X \), and if \( V \) is irreducible, so is \( X \).

4) However, \( X \) is rarely affine even if \( V \) is.

The map \( T \) is a blowup.

Start with the case: blow up of \( p = (0,0) \) in \( \mathbb{A}^2 \).

**[105]**

Idea: Remove \((0,0)\) from \( \mathbb{A}^2 \) and "rewind back to" a copy of \( \mathbb{P}^1 \) corresponding to all directions you can approach \((0,0)\).

Let \( \text{Bl}_p \mathbb{A}^2 = \tilde{\mathbb{A}}^2 = \{(x,y) \mid x \in \mathbb{A} \} \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \)

\[
\text{Bl}_p \mathbb{A}^2 = \mathbb{V}(x^i - y^j) \subseteq \mathbb{P}^1
\]

- Fiber over \((x,y) = (0,0) = (0,0) \cdot (1,1) \in \mathbb{P}^1\) is a single point.
- Fiber over \( p = (0,0) \) is \( \mathbb{V}(x^i - y^j) \in \mathbb{P}^1 \).

To "see" the blowup \( \text{Bl}_p \mathbb{A}^2 \), think how it looks in affine patches:

\( U_1 = \mathbb{A}^2 \times \mathbb{P}^1 \) \quad \( U_2 = \mathbb{A}^2 \times \mathbb{P}^1 \)

\( \tilde{\mathbb{A}}^2 \cap U_1 = \mathbb{V}(x^i(x^j - y^j)) \subseteq \mathbb{A}^3 \)

\( \tilde{\mathbb{A}}^2 \cap U_2 = \mathbb{V}(x^{ij} - y^{ij}) \subseteq \mathbb{A}^3 \)

(over affine patches)
\[ V(x^{\frac{1}{3}} - y) \subseteq A^2 \]

Note: \(B_{\mathbb{P}^2}A^2\) is smooth.

Also, \(B_{\mathbb{P}^2}A^2\) contains a dense open set \(\subseteq A^2\).

So blow-up is a birational map.

Note also that \(B_{\mathbb{P}^2}A^2 \cap B_{\mathbb{P}^2}A^2 \subseteq A^2\).

\[ V(x^{\frac{1}{3}} - y) \rightarrow A^2 \]

\[ (x, y^{\frac{1}{3}}) \rightarrow (x, y) \text{ if } x = 0 \text{ and } (x, y^{\frac{1}{3}}) \rightarrow (x, x^{\frac{1}{3}}, y^{\frac{1}{3}}) \text{ if } x \neq 0 \]

**Example**

\[ V(y^2 - x^2 - x^3) \]

\[ \pi^{-1}(c = 0) \subseteq B_{\mathbb{P}^2}A^2 \]

\[ \pi^{-1}(c = 0) \text{ is a smooth curve} \]

10/31/01 The group of \(p \in A^N\) is \(B_{\mathbb{P}^2}A^N = \{ (x, y) \mid x \in A, y \in A \} \times \mathbb{P}^{N-1} \)

where \(x \in 2 \times 2(y, x) = 0 \quad \text{and} \quad (x_1, \ldots, x_N) = 0 \)
There's a natural surjection $\text{Bl}_p A^n \rightarrow A^n$ (sometimes called $\text{blowup}$).

This is an isomorphism $\tilde{\pi} : \text{Bl}_p A^n \rightarrow A^n$. A blowup map is called a blowup map.

Inverse map $(x_0, \ldots, x_n, \xi) \mapsto (x_0, \ldots, x_n)$.

Blowing up of $V$ is a birational morphism and projective. The fiber over $p \in \mathbb{P}^{n-1}$ gives the exceptional fiber.

Now, $V \subseteq A^n$ be a closed set. The blowup of $V$ at $p$ is $\text{Bl}_p A^n$.

$$V = V(x^2y^2 - z^3) \subseteq A^3$$

$$\text{Bl}_p A^3 = E = \{(x,y,z) \in A^3 \mid (x,y,z) \neq (0,0,0)\}$$

$$\pi^{-1}(V-p) = \{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$$

Fix affine chart $U = A^3 \times A^2$

$$(x,y,z) = (x,y,z,1)$$

$$\pi^{-1}(V-p) = \{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$$

$$\text{Bl}_p U, \pi(U) = \{(x,y,z,1) \mid (x,y,z) \neq (0,0,0)\}$$

$$\pi^{-1}(V-p) = \{(x,y,z,1) \mid (x,y,z) \neq (0,0,0)\}$$

$$\text{Bl}_p V \cap U = \{(x,y,z,1) \mid (x,y,z) \neq (0,0,0)\} \subseteq A^5$$

How do we see $\xi$?

Project to $\mathbb{A}^3$ using $A^5 \rightarrow \mathbb{A}^3$

$$(x,y,z,w) \mapsto (x,y,z)$$

$$\text{Bl}_p V \cap U \cong U$$

$$\pi^{-1}(V-p) = \{(x,y,z,1) \mid (x,y,z) \neq (0,0,0)\}$$

inverse map $$(u,v,w,z) \mapsto (u,v,w)$$
Rational map

**Example**  
\[ \mathbb{P}^n \longrightarrow \mathbb{P}^{n-1} \]  
\[ [x_0 : \ldots : x_n] \longrightarrow [x_0 : \ldots : x_{n-1}] \]

Loosely a rational map is a regular map on a dense open set.

**Definition** A rational map between irreducible varieties \( X \rightarrow \mathbb{P}^n \) is an equivalence class of regular maps from dense open sets of \( X \).  
\( F = (U \rightarrow \mathbb{P}^n) \) where \( U \subseteq X \) is non-empty (dense) open subset of \( X \).  
\( F \big|_{U \cap U'} = F \big|_{U' \cap U} \) and \( \text{lim}_{u \rightarrow u'} F(u) \) is not necessarily defined.

**Example**  
\[ \mathbb{P}^2 \longrightarrow \mathbb{P}^2 \]  
\[ [x : x_1 : x_2] \longrightarrow [x : x_1 : x_2, x_3] \]  
\[ \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \]  
\[ [x : x_1 : x_2] \longrightarrow [x : x_1 : x_2, x_1 - x_2] \]

Composition \( G \circ F \) is well defined.  
\( \mathbb{P}^2 \) is not closed set of \( \mathbb{P}^2 \) where \( G \) is not defined.

**Problem** \( \pi(F) \) is closed set of \( \mathbb{P}^2 \) where \( G \) is not defined.
Rational maps

**Definition.** Two complex varieties are birationally equivalent if they are mutually inverse rational maps between them, $X \sim Y \iff \exists f : X \to Y, g : Y \to X$ such that $f \circ g = \text{identity}$ and $g \circ f = \text{identity}$. Hence $X \sim Y$ and $Y \sim X$ are identity rational maps.

**Example.** $X \approx Y \Rightarrow X \sim Y$ but not conversely.

1. $P^2 \sim P^2 - \{p\} \sim A^2, \pi^2, P^1 \times P^1$

2. $P^2 \sim P^1 \times P^1$

3. $P^1 \times P^1 \sim P^2$

**Remark.** Birational equivalence is an equivalence relation.

**Theorem.** If $X, Y$ irreducible, then $X \sim Y \iff k(x) \cong k(y)$ as extensions of $k$.

**Sketch of the proof.** (1) $X \sim Y$ then $\text{Var}(X)$ and $Y$ are affine, replacing by an even smaller affine $X \cong Y \iff k[x] \cong k[y] \iff k(x) \cong k(y)$.

(2) Conversely, $\text{Var}(X, Y)$ affine $k[x] = \prod_{i=1}^{r} k[x_i]$, $h \cong k[y] = \prod_{i=1}^{r} k[y_i]$. Assume $k \| k[y]$. Then $k[x] \cong k[y]$.

(3) By $k[x] \cong k[y]$, get a dominant map $\phi : k[x] \to k[y]$.

(4) Since $\phi$ is dominant, $\phi$ has an expression $\phi(x) = \sum_{i=1}^{r} a_i x_i^{n_i}$.
Blowing up point

Consider the rational map \( \mathbb{A}^2 \to \mathbb{P}^1 \)

\[ (x, y) \mapsto [x : y] \]

Given any rational map \( X \to Y \), the graph \( G_f \) is by definition the closure in \( X \times Y \) of the set \( \{(x, F(x)) | x \in X, F \text{ is defined} \} \subseteq X \times Y \). Note \( G_f \subseteq X \times Y \) as a closed subset.

Also, the natural projection \( \pi_f : G_f \to X \) is a projective 

\[ (x, y) \mapsto x \]

The inverse \( X \to G_f \)

\[ x \mapsto (x, F(x)) \text{ whenever } F \text{ is defined.} \]

Look at the graph of \( F \):

\[ \mathbb{A}^2 \to \mathbb{P}^1 \]

Let's look at the graph of the map

\[ \Gamma = \{(x, y) \in \mathbb{A}^2 \times \mathbb{P}^1 | (x, y, F(x)) \in \mathbb{A}^2 \times \mathbb{P}^1 \} \]

\[ \mathbb{A}^2 \to \mathbb{P}^1 \]

Now, let \( V \) be any affine irreducible variety. Take \( f_1, \ldots, f_r \) arbitrary regular functions on \( V \). Assume \( D \subseteq \{ f_1, \ldots, f_r \} \neq \emptyset \).

Consider the rational map \( V \to \mathbb{P}^r \)

\[ x \mapsto [f_1(x) : \ldots : f_r(x)] \]

This is defined outside \( V \cap \{ f_1 = \ldots = f_r \} \). The graph \( \Gamma_f \subseteq V \times \mathbb{P}^r \) is defined by

Blow up of \( V \) along the ideal \( (f_1, \ldots, f_r) \) (also called the blow up of \( V \) along the subscheme defined by \( (f_1, \ldots, f_r) \)).

Note: The map \( \Gamma_f \to V \) (Blow up map) is projective and birational, and in fact \( \Gamma_f \cap \Pi_f^{-1}(w) \to V \) where \( W \cap \{ f_1 = \ldots = f_r \} \notin V \).

[Remark] If you choose a different set of generators for the ideal \( (f_1, \ldots, f_r) \), then the resulting blowup \( \Gamma_f \) is isomorphic.

2. CAUTION! If \( f_1, \ldots, f_r \) and \( g_1, \ldots, g_s \) have the same radical, the blowups can be completely different.
For $f_1, \ldots, f_r \in k[V]$ such that the natural map \( V \to \mathbb{P}^{r-1} \)

has a smooth graph \( \Gamma_f \).

Furthermore, \( V(f_1, \ldots, f_r) \subseteq V \) is the singular set of \( V \).

Note: In this case, the natural projection onto the first factor \((\Gamma_f \subseteq V \times \mathbb{P}^{r-1})

\[ \pi \to V \]

\[ (x,y) \to x \]

is bijective (isomorphic to \( \mathbb{P}^r = \pi^{-1}(\text{Sing } V) \to V \text{-- ring } V \) and projective.

CAUTION: In general, it is virtually impossible to find the \( f_i \)’s explicitly (research area). Also, the \( f_i \)’s are not unique.

Another approach to resolving singularities of \( V \).

\( V \)-variety Hirzebruch associated an invariant \( \iota \) \( \iota \subseteq \text{Sing } V \).

You can find a smooth subvariety of \( V \), and then it up, then check that the resulting blowup is “less singular” in the sense that the invariant is smaller.

After finitely many such blowups (along “smooth centers”) the process stops at a smooth variety, which is a resolution of singularities for \( V \). Advantage of this approach: at each stage, you are doing an understandable geometric process.

**Definition.** A blowup of \( V \) along a smooth subvariety / at a smooth center / is a standard blowup, i.e., a graph \( V \to \mathbb{P}^{r-1} \)

\[ \pi \to \mathbb{P}^{r-1} \]

where \( \pi \) is smooth, a reduced ideal which defines a smooth subvariety.

In general, blowing up \((f_1, \ldots, f_r) \subseteq V)\) replaces \( V \) by some projective variety. If \((f, \ldots, f) = I(V, \omega, W) \), then \(W \subseteq V\) is a smooth subvariety, then you are taking a “projection; maximal bundle of \( W \) in \( V \).”

\[ \pi \subseteq \mathbb{A}^n \times \mathbb{P}^r \]

\[ (x_0, x_1, \ldots, x_n, 1) \]

\[ (x_0, x_1, \ldots, x_n, u) \]

\[ (x_0, x_1, \ldots, x_n, u) \]
LOCAL PARAMETERS (shaf. S 2.1)

Let $p \in V$ be a smooth point on a variety, $\dim_p V = n$.

**Definition:** Functions $u_1, \ldots, u_n \in \mathcal{O}_{p}$ are called local parameters for $V$ at $p$.

1. $\mathfrak{m} = \{ f \in \mathcal{O}_{p} : f(p) = 0 \}$
2. Their images in $\mathcal{O}_{p}/\mathfrak{m}$ form a vector space basis.

**Fact:** The elements $u_1, \ldots, u_n$ are a minimal set of generators for the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{p}$ (Nakayama's Lemma).

**Examples:**

- $\mathbb{A}^2$, $p = (0, 0)$, $1 \times y$ - local parameter $x$
- $y - x^2, x$ - local parameter $y$

**Example:**

Remember $\mathcal{O}_{p}/\mathfrak{m} \cong \mathbb{C}^n$.

- $d\nu_{u} = \sum \frac{\partial u}{\partial x_{i}} | f(x-p)$.

So, there $u \in \mathfrak{m}$ form a basis set of local parameters $\Rightarrow$

- $d\nu_{u} = d\nu_{u} = \ldots = d\nu_{u} = 0$ has trivial solution.

**Theorem:** The subvarieties of $V$ defined by $V_i = V(u_i) \subseteq V$ are smooth of dim $n-1$ in a subset of $p$ (Hence, $V$ is smooth at $p$ of dim $n$), and $u_1, \ldots, u_n$ are a local system of parameters.

All the tangent spaces $T_p V_1, \ldots, T_p V_n$ have trivial intersection.

**Example:** $p \in V$ smooth surface.

Read Sheaves. 2.1.3.

Recall: For a smooth $p \in V$, a set of $n \geq 5$ functions

- regular at $p$ (i.e., in a nbhd of $p$) is called a set of local parameters if (1) all $x_i$ vanish at $p$.

The image of $u_i \in \mathcal{O}_{p}/\mathfrak{m}^{2}$ is a basis for $\mathcal{O}_{p}/\mathfrak{m}^{2}$.
**Theorem:** If \( \{ V_1, \ldots, V_n \} \) is a set of local parameters at a smooth point \( p \) on \( V \), then

1. the hypersurfaces \( W(V_i) \subseteq V \) are smooth at \( p \) (of dimension \( n-1 \)).
2. \( T_p V = V_1 \cap \cdots \cap V_n \) is trivial (as an \( n \)-plane).

**Proof:** Assume \( \dim V_i > n-1 \). There is a surjection of local maps \( \Omega^{n-1}_V \rightarrow \Omega^{n-1}_{V_i} \)

\[ f : \Omega^{n-1}_V \rightarrow \Omega^{n-1}_{V_i} \]

\[ \dim \Omega^{n-1}_V = n \]

Hence, if \( \dim V_i = n-1 \), then \( \dim \Omega^{n-1}_V = \dim \Omega^{n-1}_{V_i} \).

\[ \Rightarrow \dim V_i = n-1 \text{ and } p \text{ is a smooth point of } V_i. \]

For (2), let \( V_i \) be a local basis. Then \( V = \bigcap_{i=1}^n V_i \) is defined by \( \pi^* V_i \). By the local nature of \( d_p \), we have \( V(d_p \Omega^{n-1}_V, d_p \Omega^{n-1}_V) = 0 \) because

\[ \frac{d}{d_p} \Omega^{n-1}_V \]

\[ \Rightarrow \frac{d}{d_p} \] (local geometry)

**Definition:** Subvariety \( Y \) of a smooth variety \( V \) intersects transversely at a point \( p \) if \( \dim \left( \pi^* (V_i \cap \cdots \cap V_n) \right) = \sum_{i=1}^n \dim \Omega^{n-1}_V - \dim \Omega^{n-1}_{V_i} \).

The theorem says that the hypersurface defined by local parameters at a smooth point on a variety intersects transversely (interest with normal crossing).

**Definition:** A local ring \( R \) is regular if its Krull dimension equals \( \dim \) of \( R \):

\[ \dim R = \text{maximal ideal of } R \text{ is always unique}. \]

(Algebraic analog of a smooth point of a variety)

**Next goal:** Fix \( Y \subseteq X \), \( X \)-irreducible, \( Y \) is closed in \( X \).

**Question:** When is \( Y \) locally defined by a single equation?

**Definition:** Functions \( f \) on a regular \( X \) in a neighborhood of \( P \) satisfy \( f = 0 \) locally if defining equations for \( Y \) at \( P \) are locally equivalent.

\[ \Rightarrow f \text{ defines an open subset of } P \text{ such that } \forall Y ( f \text{ defines } Y \text{ at } P \text{ locally}) \]
2) \( (f_1, \ldots, f_r) O_{x_P} = \prod (y) O_{x_P} \)

An obvious necessary condition on \( Y \) is that it has codimension 1 in \( X \). But this is not sufficient in general.

**Theorem** If \( V \) is a smooth irreducible variety, then every subvariety of codimension 1 in \( V \) is locally defined by a single equation.

More generally, if \( Y \) is locally defined of pure codimension 1 in \( V \) is locally defined by a single equation at any smooth point \( p \).

We've proved this when \( V = \mathbb{A}^n \), \( Y \subset V \) general \( V \), and \( V \) is smooth.

\( \mathcal{I}(Y) \subseteq \mathcal{I}(V) \), \( \mathcal{I}(Y) \mathcal{O}_p \subseteq \mathcal{O}_p \). Take any nonzero \( f \in \mathcal{I}(Y) \mathcal{O}_p \).

Assuming \( \mathcal{O}_p \) is a UFD, then when \( f \) is irreducible,

\[ \left\{ f \right\} = \mathcal{I}(Y) \subseteq \mathcal{I}(V) \rightarrow \left\{ f \right\} \mathcal{O}_p. \]

\[ \text{H/3/01} \]

**Theorem** (Step 3.1) An irreducible variety \( Y \) of codimen 1 in a variety \( V \) has a single defining equation locally in a neighborhood of a smooth point \( p \in V \).

If \( V \) is smooth, an irreducible codim 1 subvariety can be thought of locally as a hypersurface in \( V \).

\[ \text{rte} \mathcal{V} = \mathcal{U} \text{-neighbor of } p \text{ (open affin)} \subseteq V \text{ s.t. } \mathcal{Y} \subseteq \mathcal{U} \]

Caution: This does not mean \( Y \) is globally a hypersurface in \( V \), even if \( V \) is affine.

**Caution:** The theorem is not valid in general at a non-smooth point of \( V \).

**Example:** \( V = V(xy - 2z) \subseteq \mathbb{A}^3 \). This has an isolated singularity at the origin \( (0,0,0) \).

\[ W = \mathbb{N}_x \mathbb{N}_y \subseteq \mathbb{A}^2 \], by \( K \), \( \text{Ker } 
\text{mer } \) has a local

\[ d \cdot x^2 + d \cdot y^3 \]
Point of the proof: $C_p$ is a UFD, when $p$ is a smooth pt of $V$.

$$\mathcal{O}_V|_{C_p} = \frac{k[x, y, z]}{(xy - 2z)}$$

(XY = 2z, so not UFD.)

But UFD does not imply smooth point.

**Theorem**: If $X$ is smooth and $\mathbb{A}^n \dashrightarrow \mathbb{P}^n$ is a rational map for a smooth $X$, then the set of points at which $f$ fails to be defined (i.e., regular) has codimension $\geq 2$.

Recall: $\mathbb{A}$ is really an equivalence class of maps $\mathbb{A} \rightarrow X$. $f$ is regular at $p$ means that $f$ has a representative $U \rightarrow Y$ s.t. $f|_U$ is defined (i.e., is regular) at $p$.

**Corollary**: Any rational map from a smooth curve to a quasi-projective variety is actually a regular morphism, i.e., it's defined everywhere.

**Corollary**: $X$ and $Y$ smooth, $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is birationally equivalent to $Y$ $\iff$ $X$ is isomorphy to $Y$.

**Example 1**: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$:

$$[\begin{array}{c} x \\ y \\ z \end{array}] \dashrightarrow [\begin{array}{c} x \\ y : z \end{array}]$$

This map is regular everywhere except at $[1:0:0]$.

**Example 2**: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$:

$$C = w(x^2 - z^2)$$

We can then represent this map by:

$$[x: y: z] \dashrightarrow [x : y : z]$$

$C \rightarrow \mathbb{P}^1$ is regular.

**Example 3**: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$:

$$[x: y: z] \dashrightarrow [\frac{x}{z}, \frac{y}{z}]$$

This is closed. Say $E \subset E'$ is an irreducible component of codimension 1 (if $f$ is an open immersion in $\mathbb{P}^1$).

Take $p \in E$ look in a subfield of $\mathbb{P}$.

$X \dashrightarrow \mathbb{P}^n$ s.t. $E$ where $q$ is regular in a subfield of $\mathbb{P}$, wLOG.
all \( f_i \)'s are eq. at \( p \) (clearly so). Because \( \mathcal{O}_p \) is a local U.F.D., factor each \( f_i \) uniquely into irreducible polys, cancel out common factors.

WLOG, can assume the \( f_i \)'s have no common factors. Let \( (g) \) be a local defining equation for \( E_0 \) in a nbhd of \( p \).

\[
\text{Use } E_0 \ni (\{f_i \}) \quad \text{if } E_0 \ni E \ni V(\{f_i \}) \quad (\text{trans. } / \langle g \rangle)
\]

\[
\text{e.g. } 3 \quad \text{if have a common fct. for all contradiction. } \square
\]

Three kinds of classification problems in \( \mathbb{A}^n \).

1. Classify smooth varieties in \( \mathbb{P}^n \) (fixed \( N \)) up to \( \mathbb{P} \) projective equivalence.
   \( (X \equiv Y) \iff \exists \text{ a linear change of coordinates } \phi : \mathbb{P}^n \to \mathbb{P}^n \) taking
   \( X \to Y \).
   Hilbert scheme monodromy group action.

2. Classify projective varieties (smooth up to \( \mathbb{P} \)) isomorphic (being of module fun).

3. Classify \( \mathbb{P} \) toric variety, varieties up to birational equivalence.

11/12/01

**Theorem.** A rational map \( X \longrightarrow \mathbb{P}^N \) where \( X \) is smooth is regular except on a closed set of codim \( \geq 2 \).

**Note.** The target here must be (closed in) \( \mathbb{P}^N \).

**Example.** \( \mathbb{P}^2 \longrightarrow \mathbb{A}^2 \)

\[
(x, y) \mapsto \left( \frac{x}{y}, \frac{1}{z} \right) \text{ regular everywhere except at } V(z) \subset \mathbb{P}^2 \text{ (singular)}
\]

**Divisors.**

**Definition.** Let \( X \) be an irreducible variety. A prime divisor is an irreducible closed subvariety of \( \text{dim} \geq 1 \).

A divisor is a finite \( \mathbb{Z} \)-linear combination, i.e., an element of the

**Example.** On \( \mathbb{A}^n \), prime divisors are points \( \overline{p} = [1] \).

**Example.** On \( \mathbb{A}^n \), prime divisors are points \( \overline{p} = [1] \).
on $\mathbb{P}^2$, prime divisor a non-negative integer

example of divisors $2L, -17C$ where $L - V(x) \subseteq \mathbb{P}^2$

$C : W(y^2, x^3) \subseteq \mathbb{P}^2$

The divisors form a group denoted $\text{Div}(X)$ with "0" the combination \[ \sum \alpha_i \mathcal{O}_i \]

**NOTE** Every non-zero rational function on $\mathbb{A}^n$ determines a divisor, i.e., its "divisor of zeros and poles.

\[ k(\mathbb{A}^n)^* \longrightarrow \text{Div}(\mathbb{A}^n) \]

\[ \text{divisor of } f = \text{div}_1 f \]

\[ f = \frac{g(t)}{h(t)} \equiv (t-a)^{m} \cdots (t-b)^{n} \rightarrow a, [a_i]_i + [a, a_j]_i + b, [b_i]_i + \ldots + b, [b_i]_i \]

\[ \text{h.o.c.} \]

\[ \text{Note also that } \] $C \rightarrow \text{div}(1), \text{div}(f), \text{div}(f^2), \ldots \]

(homomorphism)

In general, for any $X$ smooth irreducible variety (actually doesn't need to be smooth).

If $X$ is normal $\Rightarrow$ enough. The $X$-normal means that $X$ has an affine cover $X = \bigcup U_i$, where each $k(U_i)$ is normal. For more, smooth & normal are known.

Then there's a natural map.

\[ k(X) - \text{div}(f) : k(\mathbb{A}^n)^* \longrightarrow \text{Div}(X) \]

\[ \text{which is a group homomorphism.} \]

To define this map, we'll define a function $\nu_e$ for each prime divisor $e \in X$

\[ k(\mathbb{A}^n)^* \longrightarrow \mathbb{Z} \]

\[ \nu_e(f) = \text{order of vanishing of } f \text{ along } e \]

This function will satisfy

\[ \nu_e(fg) = \nu_e(f) + \nu_e(g) \]

for $f, g \in X$, $\nu_e(f) \geq 0$, $\nu_e(g) \geq 0$, and $e$ is a component of $W(f)$

\[ \text{Use this to define } \text{div}(f) = \sum_{e \in X} \nu_e(f) \cdot e \]
For any $f$, at most finitely many $C$'s exist (prove this).

$s.t. \nu_c(f) \neq 0$.

It remains to define $\nu_c$, which we do as follows:

Fix $C$, an irreducible, connected subvariety of $X$.

On a sufficiently small affine subset $U$, $C \cap U$ is $\mathbb{A}^1$-affine.

Then $\nu_c(f) = k$.

First let's define $\nu_c(f)$ for $f$ regular on $U$.

Find $k$ s.t. $f \in \mathcal{O}^k(U)$, but $f \notin \mathcal{O}^{k+1}(U)$. Then $\nu_c(f) := k$.

Note: $\nu_c(f) < (k) \Rightarrow \nu_c(f \cdot f) = k$, so $k \geq c(m_c(C))$.

It can be shown that:

1) $\nu_c(f)$ is well-defined (indeed, $f$ has a logarithmic differential, and)$\nu_c(fg) = \nu_c(f) + \nu_c(g)$

2) $\nu_c(f+g) \geq \min(\nu_c(f), \nu_c(g))$

For $f \in k(X)^* = \text{fraction field of } k[C]$, $f = \frac{g}{h}, \ g, h \in k[U]$

$\nu_c(f) = \nu_c(g) - \nu_c(h)$

Note that the definition of $\nu_c$ can be stated as follows:

let $\mathcal{O}_X, \mathcal{O}_C$ be the "local ring of $X$ along $C$"

$$\mathcal{O}_{X,C} := \{ \frac{f}{g} \in k(X) \mid g(p) \neq 0 \text{ for some } p \in C \}$$

Equivalently: $\mathcal{O}_{X,C} = (k[C])_p$, where $p \in \text{Spec}(k[C]) \subseteq k[U]$.

where $U$ is any open affine patch of $X$, some intersection at $C$ is necessary.

The ring $\mathcal{O}_{X,C}$ is what is called an $\text{DVR}$.

One of many equivalent definitions of a $\text{DVR}$ is: that every ideal is principal.

A $\text{DVR}$ has a unique factorization into prime ideals.
Theorem on dim of \( M \)-fiber:

\[ X \rightarrow Y \text{ holomorphic, } \dim X = d \rightarrow Y \text{ char. generic fiber} \]

\[ \dim X = d \rightarrow Y \text{ regular in } X \]

**Theorem:** If \( X \) is smooth and irreducible, then a divisor \( D \) is a formal \( \mathbb{Z} \)-linear combination of codimension 1 irreducible closed subvarieties of \( X \).

\[ D = \sum n_i C_i \]

where \( C_i \) are prime divisors.

There is a canonical map:

\[ k[X]^* \rightarrow \text{Div} X \]

\[ f \mapsto \text{div} f = \sum \text{ord}_C v_C(f) C \]

where \( v_C(f) \) is the order of vanishing of \( f \) along \( C \).

**Remark:** For any irreducible codimension 1 \( C \),

\[ k[X]^* \rightarrow \mathbb{Z} \]

\[ f \mapsto v_C(f) \text{ is a valuation} \]

over the map \( v_C(f) = \min \{ v_C(f) : f \in \text{ord}_C \} \).

**Notation:** \( D \) is a divisor, \( \text{div} D = 0 \) meaning \( D = \sum n_i C_i \) with \( \sum n_i = 0 \).

\[ \text{div} f = 0 \text{ on } X \Leftrightarrow f \text{ is regular on } X \]

Lemma (2): \( \text{div} f = 0 \) at \( x \) if and only if \( f \) is \( \in \mathcal{O}_x \), where \( \mathcal{O}_x \) is the regular ring at \( x \).

**Example:** \( P^n \) is a divisor \( C = V(f) \) where \( f \) is an irreducible polynomial.

\[ D = \sum \text{div} v_i(f) \]

**Remark:** Theory of the normal cone.
\( \text{div}(f) = \sum a_i \text{V}(f_i) - \sum b_i \text{V}(g_i) \)

Define \( \text{deg}(D) = \sum a_i \cdot \text{deg}(f_i) \)

Image is the subgroup of divisors of degree 0. (Embedding of same degree)

In particular this is easy for finite sets.

\( f = \frac{x^2(x+y)}{y^2} \)

\( \text{div}(f) = 2 \text{V}(x) + \text{V}(x+y) - \text{V}(y) \)

F: blowing up to make patches

\( f = \frac{x^2(x+y)}{1-y^2} \)

\( \text{div}(f) = 2 \text{V}(x) + \text{V}(x+y) - \text{V}(1-y) \)

\( \text{deg}^2 \in \mathbb{C}^* \)

You also need worry about \( a_0 \) here at \( \infty \).

**Definition** The divisor class group of a smooth irreducible variety \( X \)

is the group \( \text{Div}(X)/\text{Pic}(X) \).

Namely, \( \text{Pic}(X) = \) the group of principal divisors of \( X \).

\( \text{Div}(X) = \sum f_i \text{V}(f_i) \)

\( \text{Pic}(X) \to \text{Div}(X) \to \text{Cl}(X) \)

\( \text{generator} = D \cdot \text{any hyperplane} \)

\( \text{Cl}(\mathbb{C}^*) = 0 \)

Prime divisor \( \text{V}(f) \) fixed.

\( D = \sum a_i \text{V}(f_i) = \sum (f_i, \ldots, f_i^n) \)

\( \text{Cl}(\mathbb{P} \times \mathbb{P}) \cong \mathbb{Z} \oplus \mathbb{Z} \)

**Definition** Two divisors \( D_1 \) and \( D_2 \) are linearly equivalent if \( D_1 - D_2 \) is a divisor of \( \text{Cl}(X) \).

\( X \) smooth, \( D \) divisor, \( D = \sum c_i \text{V}(f_i) \). Cover \( X \) by affine patches \( U_i \).

\( c_i \) is defined on \( U_i \) by \( f_i \mid_{U_i} \in \mathcal{O}_{U_i}^* \).

\( D \) on \( U_i \) a divisor \( \text{div}(f_i, \ldots, f_i^n) \)
Can think of $D$ as group $\{ u \}\cup \{ v \}\cup \{ w \}$ up to $\sim$.

Define functions $f_i$ s.t. on $U_i$, $f_i = 1$.

$\text{div}(f_i) = \text{div}(f_k)$, i.e. $\text{div}\left(\frac{f_i}{f_k}\right) = 0$ $\Rightarrow$ $\frac{f_i}{f_k}$ is regular and $\text{div}(f_i)$ is empty.

\[ P^2 \quad L = W(X), \quad L' = W(x) \]

$D = 2L - L'$ divisor in $P^2$, deg $1$, not principal.

Hence $D$ is locally principal, i.e. for a curve of $P^2$, $\mathcal{O}_D$ is trivial.

Take the standard open cover of $P^2$: $U_x, U_y, U_z$.

$\text{div}(\frac{X}{Y}) = 0$ on $U_x$,

$\text{div}(\frac{X}{Z}) = 0$ on $U_y$,

$\text{div}(\frac{Y}{Z}) = 0$ on $U_z$.

$\text{div}(\frac{X}{Y}) = 0$ on $U_x$, $U_y$.

$\text{div}(\frac{X}{Z}) = 0$ on $U_x$, $U_z$.

In particular, $\text{div}(\frac{X}{Y}) = 0$ on $U_{xy}$.

$\text{div}(\frac{X}{Z}) = 0$ on $U_{xz}$.

$\text{div}(\frac{Y}{Z}) = 0$ on $U_{yz}$.

$\text{div}(\frac{X}{Y}) - \text{div}(\frac{X}{Z}) - \text{div}(\frac{Y}{Z}) = 0$ on $U_{xy}$.

DEFINITION: given a divisor $D$, $\text{supp}(D)$ is the union of all $D_i$.

Case 1: $\{(u_x, 1), (u_y, \frac{x^2}{y}), (u_z, \frac{y^2}{z})\}$

Case 2: $\{(u_x, \frac{x^2}{z}), (u_y, \frac{y^2}{x}), (u_z, 1)\}$

Case 3: $\{(u_x, \frac{x^2}{y}), (u_y, 1), (u_z, 1)\}$

Case 4: $\{(u_x, \frac{x^2}{y}), (u_y, \frac{y^2}{z}), (u_z, \frac{z^2}{x})\}$

$\{\text{local local defining equation for } D\}$
Remark D:

On $U_i$ get $\text{div}(\frac{f_j}{f_i})$ \\
On $U_j$ get $\text{div}(\frac{f_i}{f_j})$ \\
On $U_i \cap U_j$ $\text{div}(\frac{f_j}{f_i}) = \text{div}(\frac{f_i}{f_j})$ because $\text{div}(\frac{f_j}{f_i}) - \text{div}(\frac{f_i}{f_j}) = -\text{div}(\frac{f_i}{f_j}) = 0$.

**DEFINITION.** A Cartier divisor (or "locally principal divisor") on an irreducible $X$ consists of the following data:

1) A collection of $f_i$ of

2) A (co)array of $k(U_i)$ open sets $U_i$

On each $U_i$, a rational function $f_i \in k(U_i)$ subject to

The following compatibility conditions:

- $\frac{f_j}{f_i}$ is regular, non-zero vanishing on $U_i \cap U_j$.

Equivalent to these compatibility conditions are the following:

- $\text{div}(f_i) = \text{div}(f_j)$ on $U_i \cap U_j$.
- $\text{div}(\frac{f_j}{f_i}) = 0$ on $U_i \cap U_j$.
- $\frac{f_j}{f_i}$ is regular on $U_i \cap U_j$ and $\frac{f_i}{f_j}$ is regular on $U_j \cap U_i$.

**NOTICE.** Every Cartier divisor gives rise to a Weil divisor (i.e. a geometric object described by lines) by defining on $U_i$, $\text{div}(f_i)$ and noticing that the compatibility conditions ensure the Weil classes patch together, i.e., they are equal in $U_i \cap U_j$.

Conversely, any Cartier?

If $X$ is smooth, then every Weil divisor is locally principal.

(Normal is not enough.)

**Example.** $V(y^2 - x^4) \subseteq \mathbb{A}^2$.

$D = V(y, x)$ is a Weil divisor but not locally principal

Let $\mathcal{P}(X) \subseteq \mathcal{C}(X)$

"principal" Cartier divisors

**DEFINITION.** The Picard group of $X$ ($X$ irreducible) is the group $\text{Pic}(X) = \mathcal{P}(X)$.

**NOTICE.** $\mathcal{P}(X) \subseteq \mathcal{C}(X)$.

$\mathcal{P}(x) \cong \mathcal{C}(X)$.
**IMPORTANT FACT** If $X \rightarrow Y$ is a dominant map of irreducible varieties, and $D$ is a locally principal divisor on $Y$, then there's a good notion of pull-back divisor $\varphi^*(D)$. More generally, if $\nu$ need not be dominant, we still need $\nu \in \operatorname{Supp} D$.

To see this, pick $\mathfrak{d}$ as given by data $\{ \alpha_i, f_i \}$, where $\beta_i$ is $k[Y]$ (with compatibility conditions), and $\mathfrak{d}$ affords $\nu$. We define $\varphi^*(D)$ by

$$ \varphi^*(D) = \nu \circ f_i$$

Take $\varphi^*D$ given by $\{ \varphi^{-1}(U_i), \varphi^*f_i \}$. (Check that these satisfy the same compatibility conditions as before.)

---

**EXAMPLES**

1. $C = V(y^2 - x^2) \subset \mathbb{A}^2 \setminus \{ (0,0) \} 
\text{ or } C = V(y^2 - x)$

2. $\mathbb{P}^1 \setminus \{ (0,0) \} \quad U_i = \mathbb{A}^1 \setminus \{ (0,0) \}$

$p = [1:i], i \neq 0$

Recall, if $X \rightarrow Y$ is a regular morphism of irreducible varieties, and $D$ is a locally principal divisor on $Y$ such that $\varphi(X) \setminus \nu \,(D)$ is defined as a locally principal divisor on $X$.

Let $f_i : U_i \rightarrow X$ be a regular chart.

$\varphi^*f_i = \varphi^*(x_i)$ want $\varphi^*g \circ f_i \neq 0$ and $\varphi^*h \circ f_i \neq 0$ define divisor.

What is $\varphi^*p$?

$p = \frac{1}{x-1}$ on $U_i \rightarrow \mathbb{A}^1 \subset \mathbb{P}^1$

$\varphi^*D = \nu \circ (x-1)$ on $\mathbb{A}^1$.

Let $X = \mathbb{A}^1$.

We want the new and perfect. $\operatorname{Supp} \varphi^*D = V(x-1)$ on $\mathbb{A}^1$.

Therefore $\operatorname{Supp} \varphi^*D = \{ [1:1] \} \subset \mathbb{A}^2_{\mathbb{A}^1}$.
REMARK: The map $\text{Ca.Div}_Y$ is not defined at $x$.

Because (globally) maximal divisors pull-back to finite cyclic principal divisors, we get an induced map $\text{Pic}_Y \to \text{Pic}_X$.

To compute $\text{Pic}_X$ explicitly, consider the group $\text{Pic}_X$. An ideal $\mathfrak{a}$ corresponds to the divisor $\mathfrak{a}$.

The mapping $\text{Pic}_X \to \text{Pic}_Y$ is induced by $\text{Ca.Div}_Y$.

So, for $P \in \text{Pic}_X$, let $P = \mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $\text{Pic}_X$.

Knowing $\mathfrak{a}$, we can determine $P = \mathfrak{a}$, and similarly for $\mathfrak{b}$.

Consequently, we have $P = \mathfrak{a}$.
THEOREM: For any locally principal divisor $D$ on an irreducible variety $X$, and any finite set of points $x_1, \ldots, x_n \in X$, define $D'$ such that
1. $D \sim D'$
2. $x_i \notin \text{Supp } D'$ for all $i = 1, \ldots, n$.

Theorem 2: Given $X \to Y$, to define $\text{Pic}_Y \to \text{Pic}_X$.

This works whenever the image is $(p) \subset \text{Supp} D$. Pick any point $x \in \text{Supp} D$, let $E = D - x$. We can define $p^* E$.


1. Enough to check conditions for $\text{Supp } D$ separately. For a prime divisor $D'$, assume $D$ is a prime divisor.

In a small neighborhood, let $f$ be a local defining equation of $D$. Choose $x \in D$, $f(x) = 0$. Let $E = D - \text{div}(f)$. Mapping $f$ on a section function $x$. 
(π_u : on U(ω) π_c k[U] L_k(U) = k(x))

By construction D ∼ D', x ∉ Supp D'. Look at D' on U. D'∩ U = D∩ U-supp(π_1) = 0

This completes the proof.

\[ D \sim D' \quad \text{on} \quad U \]

\[ D' \cap U = D \cap U - \text{supp}(\pi_1) = 0 \]

\[ \bullet \]

**Remark:** If D in Th. 2 is effective (all coefficients are Z ν) then ker D is in general of keeping D effective.

11/26/01 (Selj. III 3.1.4)

Divisors & Rational Maps

**Example:** \[ \mathbb{P}^2 \to \mathbb{P}^4 \]

\[ [x : y : z] \to [xy : x^2 : y^2 : z^2] \]

Write using rational functions on \( \mathbb{P}^2 \):

\[ U_2 = k[x : y : z] \setminus \{0 \} \]

\[ \frac{z}{x} \]

\[ \mathbb{P}^1 \to \mathbb{P}^4 \]

\[ [x : y : z : t : l] \to [x^2 : y^2 : z^2 : t^2 : l^2] \]

\[ \bullet \]

Look at \( f_0 \) on \( U_2 \) : \( f_0 = \frac{xz}{y} \)

\[ \mathbb{P}^2 \to \mathbb{P}^4 \]

**Note:** On the \( U_{\gamma} \) \( K_{\gamma} \)

\[ U_{\gamma} = [x : y : z] \to [y^2 : z^2 : 1 : x^2 : z^2] \]

\[ \frac{x}{y} \text{ divisor of } \frac{x^2}{y}, \text{ div}(\frac{x}{y}) \text{ on } \]

\[ U_{\gamma} \]

\[ \frac{y}{x} \text{ function in } U_{\gamma} \times U_{\gamma} \]

On \( \mathbb{P}_2 \):

\[ \frac{y^2}{x^2} \]

On \( U_2 \):

\[ \frac{x^4}{y^2} - \frac{x^2}{y} \]

\[ \frac{2x}{y} \text{ in } U_{\gamma}, \frac{1}{y} \text{ in } U_{\gamma} \]

\[ \frac{x^2}{y} \]

\[ \frac{y^2}{x^2} \]
The coordinates for \( \mathbb{P}^2 \) are given by the linear divisor \( \left\{ \binom{x}{y}, \binom{y}{z}, \binom{z}{x} \right\} \). This maps to the divisor on \( \mathbb{P}^2 \):

\[ L_1, L_2 \text{ such that } L_1 = W(x), L_2 = W(y) \]

Similarly, each coordinate is given by a divisor in a similar way.

\[ \binom{x^2, y^2, y^2}{x^2, y^2, z^2} \]

Note: \( \varphi^* H_0 = L_1 + L_2, \varphi^* H_1 = L_1 + L_2 \).

Things to notice:
1. Each coordinate is given by locally principal divisors. All are effective, all are in the same divisor class.
2. Think about hyperplane divisors on \( \mathbb{P}^4 \):
   \[ H_i = W(x_i) \]
   
   Note: \( \varphi^* H_0 = L_1 + L_2, \varphi^* H_1 = L_1 + L_2 \).

3. The locus of indeterminacy (the set of points in \( \mathbb{P}^2 \) on \( \varphi \)) is given in terms of these divisors. Denote \( \text{l.o.c.} \).

Alternate way to write \( \varphi: \mathbb{P}^2 \rightarrow \mathbb{P}^4 \):

\[ \binom{x}{y} \rightarrow \binom{x^2, y^2, x^2, y^2, x^2} \]

Locus of indeterminacy:
\[ \left( \sum \text{Supp}(D_i) \right) \]

General map:

Every rational map \( X \to \mathbb{P}^n \) is given by a set of \( n+1 \) effective (locally principal) divisors from the same linear equivalence class with the common components in their support. Also \( D_i = \varphi^* H_i \) where \( D_i \) is the \( i \)-th coordinate hyperplane of \( \mathbb{P}^n \).
RIEMANN-ROICH SPACE of a divisor

Invertible sheaf of a divisor

Fix $D$ - a divisor on $X$ irreducible.

$L(D)(X) = \{ f \in \mathcal{O}^*(X) \mid \text{div}(f + D) > 0 \}$

Another notation: $\mathcal{O}_X(D)(X)$ (Sheaf notation).

Quick example: $D = 0$, $\text{div}(1) = 0$, $\text{div}(f + D) > 0 \iff f \in \mathcal{O}_X(X)$ (regular functions).

This is a vector space over $k$.

1) $f, g \in \mathcal{O}(D)$, $a, b \in k \implies af + bg \in \mathcal{O}(D)$ (closedness).

2) $f, g \in \mathcal{O}(D)$, $\text{div}(f + g) > 0$, say $D = \text{div}(g)$. Then $(f \mathcal{O}(D))$ means $\text{div}(f) > 0$.

$\text{div}(f) = -\sum \text{supp}(f_i)$

and $f_i = f_i(x)$ for $x \in \text{supp}(f_i)$.

$\text{div}(f) > 0$ for $C^1 > 0$ as $f$ is regular.

Example: $X = \mathbb{P}^1$, $D = m \mathcal{O}_X[0]$, $m = [1:0]$.

$L(D)(X) = \{ f(x) \mid \text{deg}(f) > m \}$

Note: $\text{dim } L(D)(X) = m + 1$

Things will show:

1) If $D_1 \sim D_2$, then $L(D_1)(X) \cong L(D_2)(X)$ as vector spaces.

2) If $X$ is projective, then $L(D)(X) \cong \text{finite dimensional}$.

3) If $D = D_1 + D_2$, $X$ projective, then $L(D_1)(X)$ linearly equivalent to $L(D_2)(X)$.

4) $\mathcal{O}_X(D)$ forms a sheaf of $\mathcal{O}_X$-modules on $X$.

(i.e., for every open set $U$, $\mathcal{O}_X(D)(U)$ is a module over $\mathcal{O}_X(U)$).

In fact, it is locally free of rank $1$, i.e., invertible.)
11/26/01

**Proposition** If $D \sim D'$, then $L(D) \cong L(D')$ as $k$-vector spaces.

**Proof:** Since $D = D' + dv g$ for some $g \in k(x)^*$,

$$L(D) \cong L(D').$$

$$f \mapsto g \cdot f,$$

$$dv (f + D) \geq 0 \quad \Longleftrightarrow \quad dv (g \cdot f + D') \geq 0.$$
This makes sense provided that $f$ is uniquely defined up to scalar multiple.

Let $y = h(x) + 1$.  

\[ D' = D + c \text{div} f = D + c \text{div} y = c \text{div} \left( \frac{y}{x} \right) = c \text{div} \left( \frac{h(x)}{x} \right) = c \text{div} h(x) + 1 \]

So \[ \frac{f}{x} = \lambda \neq 0. \]

**EXAMPLE**  

\[ D = 2L - L' \]

\[ L = x^3 + xy^2 + y^3, \quad L' = x^3 + xy^2 + y^3 + 2L - L' \]

**TERMINOLOGY.**  

Let $X$ be projective. Consider a subpace $M \subset L(D)$. The associated projective space $P(M)$ is called a linear system of divisors (linearly equivalent).

The projective space $P(L(D))$ is called the complete linear system of $D$ (or the class of $D$).

**EXAMPLE**  

$D = L'$ on $\mathbb{P}^2$.

$P(L(D))$ is the complete linear system of hyperplanes in $\mathbb{P}^2$.

$M \subset L(D)$ is spanned by $\{x, y\}$.

The linear system $P(M)$ is the linear system of lines on $\mathbb{P}^2$ passing through $[0, 0, 1]$.

**Constructions:** Every rational map from an irreducible variety $X$ to $\mathbb{P}^N$ is given by a linear system.

\[ X \rightarrow \mathbb{P}^N \]

\[ x \mapsto [f_0(x) : \ldots : f_N(x)] \]

where $f \in h(x)$.

Set $D = \text{hcf} f_i + \text{div}(f_i)$. \[ \text{div}[(f_0 \ldots f_N)] = \text{hcf} \frac{f_0}{x} + \frac{f_1}{x} + \frac{f_2}{x} + \ldots + \frac{f_N}{x} \]

Set $D_i = \text{div} f_i - D$ effective, all in $\text{div}(D)$.

Set $D_0 = L_i - L_j$, $D_1 = L_i + L_j$, $D_2 = 2L_i$.
November 28, 2001

**Line bundles (Inv ch 8)**

\[ L = \{(p, l) | p \in \mathbb{P} \} \subset \mathbb{A}^2 \times \mathbb{P}^1 \]

\[ \pi_1 : \pi^{-1}(U_i) = \{(p, l) | p \in U_i \} \cong \mathbb{A}^1 \]

\[ \mathbb{P}^1 \rightarrow \pi \]

The variety \( L \) together with the map \( \pi \) is an example of a line bundle, called "tangential line bundle" on \( \mathbb{P}^1 \).

Consider the standard cover of \( \mathbb{P}^1 \):

\[ \mathbb{U}_0 \cup \mathbb{U}_1, \]

then

\[ \pi^{-1}(U_0) \to L \]

\[ \pi^{-1}(U_0) = \{(\lambda, \lambda^{-1}, \mathcal{E}(1, q)) | \lambda \neq 0 \} \]

\[ \mathbb{A}^1 = U_0 \to \mathbb{P}^1 \]

\[ U_0 \times \mathbb{K} \]

Similar for other patches.

\[ \pi^{-1}(U_0) \to \mathbb{A}^1 \]

We can think of \( L \) as "locally of the form \( \mathbb{P}^1 \times \mathbb{K} \)" and the projection \( \pi \) as "locally the natural projection \( \mathbb{P}^1 \times \mathbb{K} \to \mathbb{P}^1 \)."

Two ways of thinking of \( \pi^{-1}(U_0 \cup U_1) \) as "locally \( \mathbb{P}^1 \times \mathbb{K} \):"
A line bundle on a variety $X$ is a variety $L$ together with a surjective monomorphism $\pi: L \to X$ such that

1) Each open set $U_i$ has an isomorphism $\pi^{-1}(U_i) \to U_i \times k$ which is fiber preserving.

2) The maps $\phi_{ij}$ induce linear maps on the fiber functions.

$$\phi_{ij}: (U_i \cap U_j) \times k \to (U_i \cap U_j) \times k$$

$$(x, \lambda) \to (x, \phi_{ij}(x, \lambda))$$

The composition $\phi_{ij} \circ \phi_{ji}$ is such that $\phi_{ij}$ is an invertible regular function on $U_i \cap U_j$.

$$\pi^{-1}(U_i) = \{ (A^2 \beta_i)^T, \frac{\beta_i}{\beta_i^T} \mu \in k, \frac{\beta_i}{\beta_i^T} \in U_i \to U_i \times k \}
\left( \begin{array}{c} \frac{\beta_i}{\beta_i^T} \\ \beta_i \\ \end{array} \right)$$

$$\pi'(U_i) \to \pi'(U_i) \to U_i \times k$$

$$\left( \begin{array}{c} (\frac{\beta_i}{\beta_i^T})^T \\ \beta_i \\ \end{array} \right)$$

$$\pi^{-1}(U_i) \to \pi^{-1}(U_j)$$

$$\left( \begin{array}{c} \frac{\beta_i}{\beta_i^T} \\ \beta_i \\ \end{array} \right)$$

Same need $\frac{\beta_i}{\beta_i^T} \mu = \lambda$

1) Give a vector space structure ($\mathbf{C}$-vector) on each fiber over $x \in X$.$$
\pi^{-1}(x) \to \mathbf{C} \times k$$

2) Add 2 points $v, w \in \pi^{-1}(x)$ by $\psi_1(v, w) + \psi_1(w, v) \in \pi^{-1}(x)$.
Terminology:

- $L$ is the line bundle of a line bundle.
- The curve $U_i$ in a line bundle $L$ is called the local trivialization of the bundle.
- The $g_{ij}$'s $\in \mathbb{C}^*$ ($U_i \cap U_j$) (transition functions on $U_i \cap U_j$) are called transition functions for the bundle.

**Definition**

Let $L \to X$ be a line bundle on $X$, let $U \subseteq X$ be a non-empty open set. A section $s$ of the bundle over $U$ is a regular morphism $s$ such that the composition

$$
\begin{array}{ccc}
L & \to & L \\
\pi & \to & X
\end{array}
$$

is the identity on $U$.

Intuitively, a section $s$ on $U$ is choosing, for each $x \in U$, some vector $s(x) \in L_x \cong \mathbb{C}$.

Every line bundle has the zero section $X \to L$

$$
\begin{array}{c}
x \in X \\
\pi^{-1}(x) \to U \times \mathbb{C}
\end{array}
$$

The zero section is a global section $(U \times \mathbb{C})$.

The sections of a line bundle $L \to X$ can be denoted $\mathcal{L}$.

This is a sheaf,

$$
\mathcal{L}(U) = \text{sections of } L \to X \text{ over } U
$$

$$
V \subseteq U \\
\mathcal{L}(U) \to \mathcal{L}(V)
$$

If $U = U_i$, $s_i \in \mathcal{L}(U)$ s.t. $s_i \big|_{U_i \cap U_j} = s_j$ for $i, j \in U_i \cap U_j$, then exist $s \in \mathcal{L}(U)$ s.t. $s_i = s$. 

Trivial bundle

\[ X \times k \quad (x, \lambda) \]

\[ \downarrow \pi \quad \downarrow \phi \]

\[ X \]

Every line bundle locally looks like a trivial bundle

\[ L \xrightarrow{\pi^{-1}(U)} U \times k \]

Local trivialization

The sheaf of sections of line bundle \( L \xrightarrow{\pi} X \)

\[ U \longrightarrow \mathcal{O}_U(L) = \{ s : U \longrightarrow L \mid \pi \circ s = \text{id}_U \} \]

Note: Because \( \pi \circ s = \text{id}_U \), \( s(x) \in \mathcal{O}_x \cdot \pi^{-1}(x) \subseteq L \)

Sheaf of sections of the trivial bundle

\[ L = X \times k \quad \pi^{-1}(U) \xrightarrow{s} U \times k \quad (x, s(x)) \]

\[ X \ni \varphi \]

\[ s : \mathcal{O}_X \longrightarrow \mathcal{O}_X(L) \text{ where } s \text{ is a regular function on } U. \]

\[ \mathcal{O}_U(L) \cong \mathcal{O}_X(U) \text{ for every } U \text{ open in } X. \]

Therefore the sheaf of sections of the trivial bundle is isomorphic to \( \mathcal{O}_X \).

Say, \( L \longrightarrow X \) is not trivial bundle. Still true that \( U \times k \subseteq L \) at the bundle is trivial over \( U \).

\[ \mathcal{O}_U(L) \cong \mathcal{O}_X(U) \]

A section \( s \in \mathcal{O}_U(L) \)
$k(U) \rightarrow \mathcal{O}_X(U)

s \rightarrow \phi(s)

In general, $\mathcal{L}$ is locally (but not necessarily globally) isomorphic to $\mathcal{O}_X$. \( \forall x \in X \), $U$, s.t. $\mathcal{L}(U) \cong \mathcal{O}_X(U)$.

In general, it is called a locally-free $\mathcal{O}_X$-module of rank 1.

$L$ - an $\mathcal{O}_X$-module means that $\forall U$, $\mathcal{L}(U)$ is an $\mathcal{O}_X(U)$-module.

**NOTE** \( s_1, s_2 \in \mathcal{L}(U) \), \( s_1, s_2 \in \mathcal{L}(U) \)

\[ V : (s_1 + s_2)(x) = s_1(x) + s_2(x) \]

\[ 21 \ g \in \mathcal{O}_X(U) \rightarrow \mathcal{L}(U) \]

\[ \Rightarrow g \cdot s \in \mathcal{L}(U) \]

\[ \forall x \in \mathcal{L}(U) \rightarrow g \cdot s \]

Locally free of rank 1, \( \forall x \in X \), \( \forall U \), \( \forall s \in \mathcal{L}(U) \) is a free \( \mathcal{O}_X(U) \) of rank 1, \( \forall U \), \( \forall s \in \mathcal{L}(U) \) isomorphic to \( \mathcal{L}(U) \) isomorphic.

Fix a line bundle \( L \rightarrow X \). Fix a global section \( s : X \rightarrow L \).

Kind of word is \( s \) as a function.

For \( x \in U \), small nbhd., some \( s \in \mathcal{L}(U) \), \( s \in \mathcal{L}(U) \) is a free \( \mathcal{O}_X(U) \).

\[ \Rightarrow \text{globally, } \mathcal{L} \rightarrow \mathcal{O}_X \]

Locally-free of rank 1, \( \forall x \in X \), \( \forall U \), \( \forall s \in \mathcal{L}(U) \) is a free \( \mathcal{O}_X(U) \) of rank 1, \( \forall U \), \( \forall s \in \mathcal{L}(U) \) isomorphic to \( \mathcal{L}(U) \) isomorphic.

Evaluate \( s(x) \) doesn't make sense as an element of \( k \) (since you can't define automorphisms into \( k \)), let to say \( s(x) = 0 \) or not does make sense.

So, given \( s \in \mathcal{L}(U) \), \( x \in X \), \( s \) is a divisor (in fact effective) by taking the zero-set of \( s \) (taking into account multiplicities).

It's a locally-principal.
**Proposition**

Say $D$ and $D'$ are the effective divisors obtained as zero sets of sections $s$ and $s'$ of $\mathcal{L}(X)$, then $D \sim D'$.

**Proof:**

$D = V(s)$ (counted with multiplicity).

$D' = V(s')$.

Want $D - D' = \text{div} \left( \frac{s}{s'} \right)$.

Need: given any two global sections $s$ and $s'$ of some line bundle $\mathcal{L}$, the ratio $\frac{s}{s'}$ is a well-defined rational function on $X$.

Take $x \in X$, $U = x$, where the bundle is trivial.

\[
\mathcal{L}(U) \xrightarrow{\psi} \mathcal{O}_X(U)
\]

\[
s \rightarrow g \\
\frac{s}{s'} \rightarrow \frac{g}{g'}
\]

\[
\psi \times \chi(\mathcal{L}(U)) \xrightarrow{\chi} \mathcal{O}_X(U)
\]

\[
\psi \times \chi(\mathcal{L}(U)) \xrightarrow{\chi} \mathcal{O}_X(U)
\]

\[
\chi \times \mathcal{O}_X(U) \xrightarrow{\mathcal{O}_X(U)} \mathcal{O}_X(U)
\]

Make up class Dec 13, 1 pm (usual place)

12/3/01

The sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$.

Fix $X$, irreducible normal variety.

$D$: fixed line.
$O^c(D)$ is a sheaf

$O^c(D)(U) = \{ f \in k(U)^* \mid \operatorname{div} f + D \geq 0 \text{ on } U \}$

If $D = \Sigma a_i D_i$ then $\Delta(U) = \Sigma a_i D_i(U)$

Example: $D = -W(x)$ on $\mathbb{P}^2$, $W(x) = H$

$D = -H$

$O_{\mathbb{P}^2}(-H)(\mathbb{P}^2) = \{ f \in k(\mathbb{P}^2)^* \mid \operatorname{div} f + H \geq 0 \} = 0$

$O_{\mathbb{P}^2}(\mathbb{P}^2) = \{ f \in k(\mathbb{P}^2)^* \mid \operatorname{div} f \geq 0 \}$

$O_{\mathbb{P}^2}(-H)(U_y) = \{ f \in k(U_y)^* \mid \operatorname{div} f + (-H) \geq 0 \}$

$O_{\mathbb{P}^2}(-H)(U_y) = \{ f \in k(U_y)^* \mid \operatorname{div} f \geq 0 \}$

$O_{\mathbb{P}^2}(-H)(\mathbb{P}^2) = O_{\mathbb{P}^2}(-H)(\mathbb{P}^2)$ (not set to zero in higher degree, doesn't affect these)

Proposition: The sheaf $O^c(D)$ is a sheaf of $O_X$-modules and is locally free iff $D$ is locally principal. Furthermore, every locally free sheaf of $O_X$-modules is trivial as is of the form $O^c(D)$ for some locally principal divisor $D$.

Proof: $O^c(D)$ is an $O_X$-module. Hence $O^c(D)(U)$ is an $O_X(U)$-module where

$O_X(U)$ since $\operatorname{div}(gf) + D = \operatorname{div}(g) + \operatorname{div}(f) + D \geq 0$ on $U$.

Say $D$ is locally principal, i.e. $X$ has a cover $\{ U_i \}$ such that

on $U_i$, $D = \operatorname{div}(f_i)$ for $i = 1, 2, k(x)^*$.
\[ O_x(D)(U_i) = \{ f \in k(x)^n \mid \text{div}(f + D) \geq 0 \} = \{ f \in k(x)^n \mid \text{div}(f) \geq 0 \} = \]
\[ = \frac{1}{f} O_x(U_i) \cong O_x(U_i) \]

Finally, say \( L \) is a locally free sheaf of \( O_x \)-modules contained in \( k(x) \). There is a basis \( \{ U_i \} \) of \( X \) such that \( L(U_i) \cong O_x(U_i) \) for each \( U_i \).

Fix \( g \in L(U_i) = O_x(U_i) \)-module generator of \( L(U_i) \). This gives us the data of a Cartier divisor \( \frac{g}{f^m} \).

Also, \( \{ U_i \} \) is a Cartier divisor (Note: \( U_i \) are \( U_i \)). \( \frac{g}{f^m} \) is a Cartier divisor \( \frac{g}{f^m} \cdot g_2 \), and also \( g_2 \) is a generator of \( O \).

Define \( D \) as the divisor \( D = \text{div}(\frac{g}{f^m}) \) on \( U_i \).

Line bundles on \( X \)

- Line bundles on \( X \)
- \( L \) is a line bundle
- \( L \) is a sheaf
- \[ L \cong O_X(D) \]
- \[ D \cong \text{div}(g) \]
- \[ \text{effective} \]

\[ \text{Isomorphism of \( \text{divisor classes of Cartier divisors} \)} \]

\[ L \cong D \]

\[ \text{Isomorphism of \( \text{divisor classes of Cartier divisors} \)} \]

\[ \text{Effective divisors on } X \]

- Effective divisors on \( X \)
- \( L \) is effective
- \( L \) is a divisor
- \( L \) is effective
- \( L \) is a divisor
FUNDAMENTAL PRINCIPLE

Fix a line bundle $L = \mathcal{O}_X(1)$ on $X$, a normal projective variety.
Fix $s_0, s_1, \ldots, s_n$ global sections of $L$.
Get a map

$$X \longrightarrow \mathbb{P}^N$$

Regular map away from common zero set.

$$x \longmapsto \left[ s_0(x), s_1(x), \ldots, s_n(x) \right]$$

This means: on the subset $U_i$ of $X$ we have

$$U_i \longrightarrow \mathbb{P}^N$$

$$U_i \longmapsto \left[ \frac{s_0}{s_i}, \frac{s_1}{s_i}, \ldots, \frac{s_n}{s_i} \right]$$

We showed last time that $\frac{s_i}{s_j} \rightarrow 1$ is a well-defined rational function on $X$.

Dual of tautological line bundle called hyperplane bundle.

$(\mathcal{O}(1)_{\mathbb{P}^2})$ where $H = H(1)$ on $\mathbb{P}^2$.

$\mathcal{O}(1)_{\mathbb{P}^2}(H)$

Its global sections:

$$H^0(\mathbb{P}^2, \mathcal{O}(1)) = \mathbb{C}[x_0, x_1, x_2]$$

$$\mathcal{O}(1) \cong \mathcal{O}(1)_{\mathbb{P}^2}(H)$$

Fix a basis for $H^0(\mathbb{P}^2, \mathcal{O}(1))$:

$$\frac{x_0}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0}$$

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^1$$

$$\left[ x_0 : x_1 : x_2 \right] \longrightarrow \left[ 1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} \right]$$
\( \mathcal{O}_P(2H)(P^3) = \{ (c_k(P)^r) \} \quad 2\nu f + 2H > 0 \}
\[ F \quad \text{by} \quad 2 \]

\[ \text{Lemma } x_0, x_1, x_2 \]

Map
\[ P^2 \rightarrow P^5 \]
\[ \left[ x_0 : x_1 : x_2 \right] \rightarrow \left[ x_0^2 : x_1^2 : x_2^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 \right] \]
regular everywhere (Veronese)

12/7/01: Special kinds of line bundles

\[ X \text{ projective variety, } L \text{ line bundle (thick sheaf of sections \( L \))} \]

\[ D \text{ divisor class } D, \quad L = \mathcal{O}(D) \]

The space of global sections of \( L \) (Bernauer rank-see \( \mathcal{L}(D) \)) is a finite line bundle space.

Fix a set of generators \( s_0, \ldots, s_r \) (i.e., a basis)

Get a map to projective space
\[ X \twoheadrightarrow \mathbb{P}^r \]
\[ \left[ s_0(x) : \ldots : s_r(x) \right] \]

**Definition**

1. \( L \) is globally generated if this map is regular. (equivalent way to define \( L \) on \( X \), \( \Gamma \) - global section of \( L \longrightarrow \mathbb{X} \))

2. \( L \) is very ample if this map is an embedding (Nemomorphism into its image)

**Definition**

\( L \) is ample if \( L^N \otimes \mathcal{O}_X \) is very ample for all \( N \)-ample \( N \)-large enough.

\[ L \otimes \mathcal{O}_X \]
\[ L^N \otimes \mathcal{O}_X \]

\( N \)-ample \( \mathcal{O}_X \)

\( N \)-large enough

\( \mathcal{O}_X \)

\( \mathcal{O}_X \) on \( X \)

\( \mathcal{O}_X \) on \( X \)

\( \mathcal{O}_X \) on \( X \)
Many notions of degree

1) Degree of a hypersurface $\mathbb{P}^N \ni X = V(F) \subseteq \mathbb{P}^N$, \(\deg X = \deg F\).

2) Degree of a divisor on $\mathbb{P}^N$, \(D = \sum a_i W(E_i)\), \(\deg D = \sum a_i \deg E_i\).

3) Degree of an irreducible subvariety of $\mathbb{P}^N$ (later).

4) Degree of a divisor on a curve, \(D = \sum a_i p_i\), \(\deg D = \sum a_i p_i^{-1}\).

Comparison 2 & 4: If $X = \mathbb{P}^1$, \(D = \sum a_i W(E_i)\).

5) Degree of a map between varieties.

I. Degree of a map

\[ X \rightarrow^f Y \] dominant map of irreducible varieties of same dimension.

**Definition:** The degree of $f$ is the degree of the field extension $k(Y) \subset f^* k(X)$.

**Example:**
\[
\begin{align*}
\mathbb{A}^1 & \rightarrow \mathbb{A}^1 \\
 t & \rightarrow t^n \\
k[t] & \subset k[t] \\
k(t) & \cong k(t)^N, \deg N = n \\
k^N & \leftarrow t
\end{align*}
\]
Theorem. If \( X \to Y \) is a finite map of irreducible, smooth varieties, and \( p \in Y \), then
\[
\# \text{if}^{-1}(p) \leq \deg f
\]
Furthermore, this inequality holds only on a closed set

\[X' \to Y' \text{ is a finite map of irreducible, smooth varieties, and } p' \in Y', \]

\[
\# \text{if}^{-1}(p') \leq \deg f
\]

This closed subset is called the ramification locus of \( f \).

Theorem (Shaf. III, §2.1). If \( X \to Y \) is a surjective regular map of smooth projective curves, then \( \forall p \in Y, \)
\[
\deg f = \deg (f^* p)
\]

Example:
\[
\begin{array}{c}
\mathbb{P}^1 \\
\text{f}
\end{array} \to \begin{array}{c}
\mathbb{P}^1
\end{array}
\]
case \( k = 0 \)
\[
[x, y] \to [x^2, y^2]
\]
\[
p = [a, b] \in \mathbb{P}^1 \times \mathbb{P}^1
\]
\[
p \in \frac{b}{a} \overline{x^1} + \frac{y}{x}
\]

Example of \( p \in \text{Ram}(f) \)
\[
\frac{b}{a} \overline{x^1} + \frac{y}{x}
\]
\[
N(1, 1) \leq p \to \frac{1}{a} \overline{x^1} \leq p \to W(\frac{1}{a})
\]
\[
N(0, 1) \leq p \to \frac{1}{a} \overline{x^1} \leq p \to W(\frac{1}{a})
\]
\[
N(0, 1) \leq p \to \frac{1}{a} \overline{x^1} \leq p \to W(\frac{1}{a})
\]
\[
N(0, 1) \leq p \to \frac{1}{a} \overline{x^1} \leq p \to W(\frac{1}{a})
\]
\[
N(0, 1) \leq p \to \frac{1}{a} \overline{x^1} \leq p \to W(\frac{1}{a})
\]

Corollary. The degree of a principal divisor on a smooth projective curve is 1.
CAUTION: The converse is false, except when curve $= \mathbb{P}^1$.

Proof of the equality $D = \text{div} f$

Want $\deg D = 0$, $X \to \mathbb{P}^1$.

$\text{div} f = (\text{zeros of } f) - (\text{poles of } f)$

$\text{div}(f \cdot \omega_1) = \text{div}(f \cdot \omega_2)$

$\deg(\text{div } f) = \deg(f \cdot \omega_1) - \deg(f \cdot \omega_2) = 0$

... as usual by the lemma.

In fact, the divisor $D$ of degree 0 can be viewed as a smooth projective curve in a variety, called the *Jacobian* of $X$.

In fact, this variety has a natural group structure, called an *abelian variety*.

Example $X \cong W(E_3) \subset \mathbb{P}^2$ smooth (called an *elliptic curve*).

Then divisor classes of degree 0 on a 1-1 correspondence with points.

12/10/01

Degree of an irreducible subvariety $X$ in $\mathbb{P}^N$ (deg. vanishing 3)

$X \subset \mathbb{P}^N$

**Definition**

Deg of $X$ is the codimension of $X$ (finite) set of $\mathbb{P}^N$, where

$L$ is a generic linear subspace of codimension equal to the dimension $X$.

This makes sense because of the following facts:

**Theorem** For any $X \subset \mathbb{P}^N$ consider the Grassmannian $G$ of codim $d$ linear spans in $\mathbb{P}^N$. G $\subset \mathbb{P}^N$.
There's a proper closed subset of $G$ consisting of linear spaces $L$ s.t. \[ \dim(L + X) > 0 \]

So there's an open set in $G$ consisting of $L$, where $\#(X + L)$ is maximal & finite. This maximum number is the degree.

**Example**

$X \subseteq \mathbb{P}^2$

$W(x^2 - y^2)$

Target line intersect only once.

There's also alternative approach in which we assign intersection multiplicities to each $p$ in $X + L$.

**Proposition**

If $X = V(F) \subseteq \mathbb{P}^n$

Then $\deg X$ as a hypersurface in $\mathbb{P}^n$ (degree in some $\mathbb{P}^1$)

\[ \deg X \text{ (a sum of 3)} \]

**Proof:** Take a general line in $\mathbb{P}^n$. Want to count $\#(X + L)$

$X + L = V(F|_L) \subseteq L = \mathbb{P}^1$

$p(L)$, $p(F)$ is homogeneous polynomials of 2 variables.

After changing coordinates, split completely with linear forms.
**PROPOSITION**  
\[ \text{If } \lambda = 1, \text{ is a smooth curve then } \deg X = \deg H \]

\[ \text{where } H \text{ is the hyperplane divisor on } X \text{ (i.e., } H = *H, \text{ and } X_0 \in H, \]

and \(H')\) is any hyperplane in \(\mathbb{P}^n\), \(H'\) not containing \(X\).

\[ \text{Proof:}\]  
\[ \forall k \in \mathbb{N}, \quad \text{note } i^*H' = H' \cdot X \text{ is a degree } k \text{ multiplicity.} \]

\[ \deg H = \sum a_i. \quad \text{Note: The degree of } H \text{ is independent of which hyperplane } H' \text{ is taken, as long as } X \notin H' \]

And, for a randomly chosen one, all \(a_i = 1\).

If all \(a_i = 1\), then \(\deg H = \#(X \cdot H')\).

\[ X \cdot H' = \text{same number} \]

\[ \text{number of hyperplanes.} \]

\[ \text{12/12/01} \]

**BEZOUT'S THM**

If \(C\) and \(D\) are curves in \(\mathbb{P}^2\) of degrees \(a\) and \(b\) respectively, and \(C\) and \(D\) have no common components, then \(\#(C \cdot D) = \deg(C \cdot D)\).

with equality if \(C\) and \(D\) are in general position with each other, or \(= 0\) if \(C\) or \(D\) are multiple.

In general, \(X \subset \mathbb{P}^n\) is smooth curve in \(\mathbb{P}^n\). Fix \(k[x_0, \ldots, x_n]\) homog.

say degree \(e, f, \ldots, t\). \(F\) doesn't vanish everywhere on \(X\).

The divisor \(\text{div}_X(F)\) on \(\mathbb{P}^n\) can be pulled back \(\phi\) to \(X\), to get a divisor \(\phi^*\text{div}_X(F)\).

The degree of this divisor, \(\deg D\) is called the intersection \# of \(X\) with \(\text{div}_X(F) = \text{ident} X \cdot \text{div}_X(F)\).

**NOTE**

This number \(\deg D\) depends only on the divisor class of \(D\).

So, it depends only on the degree of \(F\). As \(F\) is a different degree \(e, \deg \phi^*\text{div}_X(F) = \text{div}(E)\), on \(\mathbb{P}^n\), \(\phi\) is no longer linear.
$X \cdot \text{div}(F) = X \cdot \text{div}(L^e) = X \cdot e \cdot \text{div} L = e \cdot (X \cdot \text{div} L) = e \cdot \deg X$

We just showed that $X \cdot \text{div}(F) = (\deg X) \cdot (\deg F)$

Now, let's repeat our argument with $\mathcal{C} = X$ and $\mathcal{D} = \text{div}(F)$

\[ \#(\mathcal{C} \cap \mathcal{D}) = \mathcal{C} \cdot \text{div}(F) = (\deg F) \]

Hilbert polynomial

$X \subseteq \mathbb{P}^N$ projective irreducible

$\Pi(X) \subseteq k[x_0, \ldots, x_N]$

Homogeneous coordinate ring $\frac{k[x_0, \ldots, x_N]}{\Pi(X)} = \mathbb{R}$

$\mathbb{N}$-graded ring $\mathbb{R} = R_0 \oplus R_1 \oplus \ldots \oplus R_\mathbb{N}$

Consider the Hilbert function of $X \subseteq \mathbb{P}^N$

$\Pi(N) \rightarrow N$

$m \mapsto \text{dim}_k [R_m]$

**THEOREM** For large $m$, the Hilbert function agrees with a polynomial, a polynomial called the Hilbert polynomial of $X \subseteq \mathbb{P}^N$. 
**NOTE** If \( X \subseteq \mathbb{P}^N \) and \( Y \subseteq \mathbb{P}^M \) and \( X \cong Y \) by a projective change of coordinates, then they both have the same Hilbert polynomial.

**EXAMPLE** \( X = \mathbb{P}^N \), \( R = k[x_0, \ldots, x_N] \)

\[
\dim R_m = \binom{N+m}{m} = \frac{(m+N)(m+N-1) \cdots (m+1)}{m!} = \frac{m^N}{N!} \\
\text{polynomial in } m.
\]

**Basic Facts:** Write Hilbert poly \( P_X(m) = \frac{e_0 m^d + e_1 m^{d-1} + \cdots + e_d}{d! (d-1)!} \)

1. \( \deg \text{deg } P_X = \dim X \)

2. **Leading coefficient:** \( d! e_0 = \deg X \) (as a subvariety of \( \mathbb{P}^N \))

3. The other coefficients are other projective invariants, computable in various intersection numbers.

4. \( P_X \) is a projective invariant but not an isomorphism invariant.

\( (X \cong Y \Rightarrow P_X = P_Y) \) unless \( X \) and \( Y \) are isomorphic, in which case the \( e_i \)s are the coefficients of a linear change of coordinates in \( \mathbb{P}^N \).

(same for all the coefficients)

5. The question: "At what point does the Hilbert function begin to resemble the Hilbert poly?" is answered by various vanishing of cohomology for line bundles on \( X \).

**FIN."**

6. Fix a polynomial \( P \) consider the set of all subvarieties of \( \mathbb{P}^N \) with Hilbert polynomial \( P \). This set is actually finite a natural geometric object—a scheme, called the Hilbert scheme: \( \text{Hilb}_P(\mathbb{P}^N) \).

**EXAMPLE** \( C \subset \mathbb{P}^N \)

\[
P \longrightarrow \mathbb{P}^2 \begin{array}{c}
(x, y) \rightarrow \left( x, x^2, y^2, y^4, y^6 \right)
\end{array}
\]

\( R \subset k[C] : k[x, x^2, y, y^2, y^4, y^6] \subset k[x, y] \)

\[
\dim R_m = \dim \left[ k[x, y] \right]_{m-\deg P}.
\]
$\mathcal{P}_C(m) = N \cdot m + 1$

degree of $C \cong \mathbb{Z} \times m \cdot N$