These notes are the second term of a course on class field theory. The first term covered local class field theory. As such we will focus on the analytic version of global class field theory and derive the ideal version of the results from there. This course will focus more on the statement of the theorems and applications are less on the proof of the main theorem. One should have a knowledge of local fields, local class field theory, and basic algebraic number theory before attempting to study these notes. There will undoubtedly be errors in these notes, so please use them at your own risk. We make the simplifying assumption that our global and local fields all have char 0.

Chapter 1: Adeles and ideles:

The formulation of the results of global class field theory in terms of ideles is in most natural framework.
when one has a background in local class field theory.

This framework also provides the best framework to understand the conjectural framework of the Langlands correspondence of which class field theory is the most basic case. This first chapter will contain the basic definitions we will need.

Section 1.1: Article:

Let \( K \) be a global field, i.e., for us this will be a finite extension of \( 
\mathbb{Q} \). Let \( M_K \) be the set of places of \( K \). We saw in local class field theory that \( M_K \) is in bijection with the set of normalized valuations of \( K \).

Let \( v \in M_K \) and let \( K_v \) be the completion of \( K \) wrt \( v \). (\( K_v \) is a local field of course!) Let

\[
\mathcal{X}_K = \prod_{v \in M_K} K_v.
\]

Observe that

\( K \hookrightarrow \mathcal{X}_K \)
with the maps

\[ x \mapsto (x, x, \ldots, x, \ldots) \]

using that \( K = K_v \) for each place \( v \in \text{Mk} \). For

\( x \in K_v \) one has \( x \in \mathbb{Z}_v \) for almost all \( v \in \text{Mk} \). For

example, if \( K = \mathbb{Q} \), then in the statement that \( \mathbb{Q} = \mathbb{Q} \)

satisfies \( \left| \frac{a}{b} \right|_v = 1 \) for almost all \( p \).

**Def.** The **idele** of \( K \) is the set

\[ \mathbb{A}_K = \bigoplus_v (x_v) \in \bigoplus_v \mathbb{K}_v : x_v + \mathbb{Z}_v \text{ a.e. } v \]

(a.e. stands for almost every, which in this context
means all but finitely many)

The **ideal** of \( K \) is the set

\[ \mathcal{I}_K = \left\{ (x_v) \in \prod_v \mathbb{K}_v : x_v + \mathbb{Z}_v \text{ a.e. } v \right\} \]

Observe that the idele can also often written as \( \mathbb{A}_K \). One
should check that the notation \( \mathbb{A}_K \) agrees with the notation
of writing \( \mathbb{R}^\mathbb{N} \) for the unit of a ring \( \mathbb{R} \). Namely,
first one should observe that \( \mathbb{A}_K \) is naturally a ring
and $M^x_k$ is naturally a group. One should also observe that

$$K ightarrow M^x_k$$

and

$$K^x ightarrow M^x_k.$$  

This is immediate from the fact observed above that for $x \in K$, $x \in U_v$ for a.e. $v \in M_k$.

We now make $M^x_k$ and $M^x_k$ into topological spaces.

Define the sets

$$U = \prod_{v \in M_k} U_v$$

where $U_v \in K_v$ open (with the topology given by $v$) and $U_v = O_v$ for a.e. $v \in M_k$ as a basis for the topology on $M^x_k$.

**Exercise:** Check that $M^x_k$ is a topological ring with this topology.

We can take a basis

$$U = \prod_{v \in M_k} U_v$$
where $U, \emptyset \in \mathfrak{U}$ is open and $U - \emptyset$

for a.e. $\nu \in M_K$ as a basis for the topology on $\mathfrak{A}_K$.

Exercise: Check this makes $\mathfrak{A}_K$ into a topological group.

For the adele to be interesting, it is necessary that for a
finite separable extension $L/K$ that the rings $\mathfrak{A}_L$ and
$\mathfrak{A}_K$ be related. The following lemma gives this relation.

Lemma 1.1: Let $L/K$ be a finite separable extension.

There is a topological and algebraic isomorphism

$$
L \otimes_K A_K \rightarrow A_L
$$

such that

$$
L \otimes_K B \in L \otimes_K A_K
$$

gets identified with $L \subseteq A_L$ via the map

$$
\alpha \otimes \beta \rightarrow \alpha \beta.
$$

(Observe that $L \otimes_K \mathfrak{A}_K^n$ when $n = [L : K]$, and so

$$
L \otimes_K A_K = A_K^n
$$

as $K$-vector spaces, and so

this gives the topology on $L \otimes_K A_K$.)

The isomorphism $L \otimes_K A_K \rightarrow A_L$ is given by

...
\[ \alpha \otimes (x_v) \rightarrow (y_v) \]

where \( y_v = \alpha x_v \) if \( w \mid v \).

**Proof:** We have

\[
\begin{align*}
L \otimes_K \mathbb{X}_K &= L \otimes_K \mathbb{M}_K \\
&= \prod_{w \in M_K} (L \otimes_K K_w) \\
&= \prod_{w \in M_K} \prod_{w' \in M_K} L_{w'v} \\
&= \prod_{w \in M} L_w \\
&= \mathbb{X}_L.
\end{align*}
\]

where the isomorphism is given by \( \alpha \otimes (x_v) \rightarrow (y_v) \)
where \( y_v = \alpha x_v \) if \( w \mid v \). (We proved this isomorphism last term.) From this it is clear that

\[ L \otimes_K \mathbb{M}_K \rightarrow \mathbb{M}_L. \]

**Claim:** There exist a finite set of places \( S_0 \subseteq M_K \) such that \( \mathfrak{p} \in S_0 \), then

\[
\mathbb{O}_L \otimes_{\mathbb{O}_K} \mathbb{O}_w = \prod_{w \in M_K} \mathbb{O}_w
\]

\[ \alpha \otimes \mathfrak{p} \rightarrow (\alpha \mathfrak{p}). \]

**Proof:** Let \( x_1, x_2 \in K \) be \( \mathfrak{p} \)-units. We can choose

choose \( x_1 \in \mathbb{O}_L \) by clearing the denominators, if necessary.
Let \( \Delta : A_{L_K} \rightarrow (x_1, \ldots, x_n) \). We know from algebraic geometry theory that \( \Delta \neq 0 \). Thus, if \( \mathcal{S}_0 \subset M_K \) a finite set s.t \( \| \Delta \|_{L_K} = 1 \) \( \forall \mathcal{S}_0 \). We have:

\[
\Delta \cdot \prod_{w \in M_K} \mathcal{O}_w = \sum_{w_1 \in \mathcal{O}_{L_K}} \mathcal{O}_{L_K} \subset \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L = \prod_{w \in M_K} \mathcal{O}_w.
\]

\( \mathcal{O}_L \rightarrow (x_1, \ldots, x_n) \)

(Check the containment as an exercise.)

For \( \forall \mathcal{S}_0 \), we have \( \| \Delta \|_{L_K} = 1 \) and so \( \| \Delta \|_{L_K} = 1 \) and thus \( \Delta = \mathcal{O}_L \) for all \( w \in M_K \). Thus,

\[
\Delta \cdot \prod_{w \in M_K} \mathcal{O}_w = \prod_{w \in M_K} \mathcal{O}_w.
\]

This gives that the above containment is an equality and

\[
\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L = \prod_{w \in M_K} \mathcal{O}_w \quad \text{as claimed.}
\]

We have

\[
L \otimes_{\mathcal{O}_K} \mathcal{O}_K = \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L / \mathcal{O}_K \quad \text{(factor out } \frac{1}{x_i} \text{ from } (x_i) \otimes_{\mathcal{O}_K} \mathcal{O}_K \text{ for } x \in L) \]

\[
= \bigcup_{\mathcal{S}_0 \subset M_K} \left( \mathcal{O}_L \otimes_{\mathcal{O}_K} \left( \prod_{w \in M_K \setminus \mathcal{S}_0} \mathcal{O}_w \right) \right)
\]

\[
= \bigcup_{\mathcal{S}_0 \subset M_K} \left( \prod_{w \in M_K \setminus \mathcal{S}_0} \mathcal{O}_w \right) \quad \text{(by claim)}
\]

\[
= \mathcal{A}_K.
\]

Note we are just taking unions over the open set containing \( \mathcal{S}_0 \).
which is how we obtain \([A]_k\) in \([A]_\omega\) from the unions given.

This gives the algebraic isomorphism, and it is not difficult from this to see it is also a topological isomorphism.

The following topological properties of \([A]_k\) will be important in our application.

**Theorem 1.3:** Let \(K\) be a number field.

1. \(K\) is discrete in \([A]_k\);
2. \([A]_k/K\) is compact in the quotient topology.

**Proof:** The previous lemma allows us to identify \([A]_k\) with \(K \otimes_q A_q\). Let \(n = [K:Q]\) and \(x_1, \ldots, x_n\) a \(Q\)-basis of \(K\). We have

\[
[A]_k = K \otimes_q A_q \cong \prod_{i=1}^n [A]_q
\]

\[
(\sum a_i x_i) \otimes x \to (a; x).
\]

We also have

\[
K = \prod_{i=1}^n Q.
\]

Thus,

\[
[A]_k/K \cong \prod_{i=1}^n [A]_q/Q.
\]

This shows it is sufficient to prove the result for \(k = \omega\).
(i) It is enough to find a nbhd of \( 0 \) in \( \mathbb{A}_\mathbb{Q} \) containing any other elements of \( \mathbb{Q} \). Let 

\[
U = \{ (x, y) \in \mathbb{A}_\mathbb{Q} : |x| < 1, |y|_p < 1 \text{ for all } p \}\n\]

\[= (-1,1) \times \prod_p \mathbb{Z}_p.\]

This is clearly an open set in \( \mathbb{A}_\mathbb{Q} \) containing \( 0 \).

Assume \( f \in \mathbb{Q} \), \( r > 0 \) s.t. \( r \notin U \). Let \( c = \frac{1}{r^2} \).

with gcd \( (r, c) = 1 \). Again \( f \in U \), \( 1/|x|_p < 1 \text{ for all } p \) and \( -c \in \mathbb{Z} \). However, we also have \( 1/|y| < 1 \) and \( -c \in \mathbb{Z} \). Thus \( U \cap \mathbb{Q} = \emptyset \) and \( \mathbb{Q} \) is closed in \( \mathbb{A}_\mathbb{Q} \).

(ii) It is enough to show that there is a compact set \( W \subset \mathbb{A}_\mathbb{Q} \) s.t. \( W \) surjects onto \( \mathbb{A}_\mathbb{Q} / \mathbb{Q} \). Let 

\[W = \{ (x, y) \in \mathbb{A}_\mathbb{Q} : |x| < \frac{1}{2}, |y|_p < 1 \text{ for all } p \}\n\]

\[= [-\frac{1}{2}, \frac{1}{2}] \times \prod_p \mathbb{Z}_p.\]

Note that \( W \) is a compact set. \( \mathbb{Q} \) is the product of compact sets, and so is itself compact. (Check this on an exercise to make sure you are understanding the topology here.)

Claim: \( \mathbb{Q} + W = \mathbb{A}_\mathbb{Q} \).

**Proof:** Let \( x = (x, y) \in \mathbb{A}_\mathbb{Q} \) and let \( S = \{ s \in \mathbb{A}_\mathbb{Q} : x + s \in W \} \).
By definition, $\mathcal{S}$ is a finite set. Let $v \in \mathcal{S}$ be a finite place. Write

$$x_v = \sum_{n=0}^{\infty} a_v p^n = \frac{a_{-M}}{p^M} + \ldots + \frac{a_{-1}}{p} + a_0 + a_1 p + \ldots$$

Observe that if we set

$$y_v = -\frac{a_v}{p^v} - \left(\frac{a_{-M}}{p^M} + \ldots + \frac{a_{-1}}{p}\right),$$

then $x_v + y_v \in \mathbb{Z}_p$ and $y_v \in \mathbb{G}$.

Let $x' = x + \sum_{v \in \mathcal{S}} y_v = (x_v')$. Then $x' \in \mathbb{Z}_p$ for every prime $p$. Thus,

$$x' \in \mathbb{R} \times \prod_{p} \mathbb{Z}_p.$$ 

Let $y_\infty \in \mathbb{Z}$ be such that $x_\infty + y_\infty \in [-\frac{1}{2}, \frac{1}{2})$.

Then $x + y_\infty = x + y_v + \sum_{v \in \mathcal{S}} y_v \in \mathbb{W}$. Thus

$$x' \in \mathbb{G}$$

shows the claim is true and thus completes the proof of the theorem. \qed

Let $\mathcal{S} \subseteq \mathbb{M}_K$ be a finite set. We will want to be able to avoid finitely many primes in our results and applications. This is a common phenomenon in
number theory when one often avoid archimedean
places and places that ramify. Let

\[ \mathcal{A}_K^S = \{ (x_v) \in \prod_{v \in \mathcal{S}} K_v : x_v \in A_v \text{ a.e. } v \}. \]

Exercise: \[ \mathcal{A}_K = \mathcal{A}_K^S \times \prod_{v \in \mathcal{S}} K_v. \]

We will require the notion of Haar measure to further our study.

For those who do not have any experience with Haar measure,

it is safe to think of a measure as a function that assigns

a size to a collection of sets called measurable sets. It

is also to just think of this as some function measuring the

size of sets behaving as you would expect. For those who are

familiar with Lebesgue measure on \( \mathbb{R} \), a Haar measure is a

generalization of the result to other settings. Any introductory

book on harmonic analysis will have a discussion of Haar measure

for those interested in seeing the details.

Our goal is to prove the following theorem:

\[
\text{Theorem 1.3 (Strong Approximation): } \text{ If } S \neq \emptyset, \text{ then } K \subseteq \mathcal{A}_K^S \text{ is dense.}
\]
Note that what this is saying is that if we remove finitely many places from the ring, we can approximate elements by elements in $K$.

**Fact:** The ring $M_K$ has a Haar measure $\mu$ such that

$$\mu(xS) = \prod_v \mu(x,1,v) \mu(1).$$

The idea for the case $K = \mathbb{Q}$ is as follows. For $v = \infty$, we take the Haar measure on $\mathbb{R}$ to be the usual Lebesgue measure.

For $v = p$, we take the Haar measure on $\mathbb{Q}_p$ to be normalized so that $\mu_1(\mathbb{Q}_p) = 1$. For a finite set $S \subseteq M_K$ with $0 \notin S$, we set

$$\mathbb{X}_S = \prod_{v \in S} K_v \times \prod_{p \notin S} \mathbb{Q}_p.$$

Then we have a well-defined product measure on $\mathbb{X}_S$ for any such $S$. If $S \subseteq S'$, then $\mu_S |_{\mathbb{X}_S'} = \mu_{S'}$ and so we can form the measure $\mu = \lim S \mathbb{X}_S$. This gives the Haar measure on $M_\mathbb{Q}$ since $M_\mathbb{Q} = \bigcup_S \mathbb{X}_S$.

**Theorem 1.4:** $\mu(M_K/K) < \infty$.

**Proof:** We have shown $M_K/K$ is compact and the measure
of a compact set is always finite for any Haar measure.

We now present the two results needed to prove strong
approximation.

**Lemma 1.5:** There is a constant \( c > 0 \), depending only on \( K \), so if
\[ X = (x_0) \in \mathcal{M}_K \] satisfies \( \prod_{v \in \mathcal{M}_K} \| x_{0,v} \|_v > c \), then \( \exists \alpha \in \text{marg}
\alpha \in K \) s.t. \( \| x_{\nu} \|_v \leq \| x_{0,v} \|_v \) for all \( \nu \in \mathcal{M}_K \).

**Proof:** Let \( c_0 = \mu_k(K) \) and \( c_1 = \mu_k \left( \bigcup_{v \in \mathcal{M}_K} \{ x_{0,v} : \| x_{0,v} \|_v \leq 1 \} \right) \).

Let \( c = \frac{c_0}{c_1} \) (it is clear that \( c_1 \neq 0 \)). For any \( x = (x_0) \in \mathcal{M}_K \),
\[ T_x = \left\{ (x,v) \in \mathcal{M}_K : \| x_v \|_v \leq \frac{1}{c} \| x_{0,v} \|_v \text{ if } v \in \mathcal{M}_K \right\} \text{ otherwise.} \]

Then we see
\[ \mu(T_x) = c_0 \prod_{v \in \mathcal{M}_K} \| x_{0,v} \|_v > c_0. \]

Thus, \( \exists t', t'' \in T_x \) s.t. \( 0 \neq t' - t'' \in K \).

Let \( \alpha = t' - t'' \in K \). Then
\[ \| x_{\nu} \|_v = \| t' - t'' \|_v \leq \| x_{0,v} \|_v. \]

**Corollary 1.6:** Let \( \nu \in \mathcal{M}_K \) be a fixed place. Suppose that for each \( \nu \neq \nu_0 \) we
have \( S_\nu \in \mathbb{R}_+^+ \) with \( S_\nu = 1 \) for \( \nu_0 \). Then \( \exists \alpha \in \text{marg} \setminus K \) s.t.

...
Proof: Choose \( x = (x_v) \in M_\mathbb{K} \) such that \( 0 < \| x \|_v \leq \delta_v \) for every \( v \neq v_0 \) and \( \prod_{v \neq v_0} \| x \|_v > C \) with \( C \) given as in Lemma 1.5. Now just apply Lemma 1.5.

Exercise: Using the compactness of \( K \setminus \{ 0 \} \), show that there exist \( S_v \in \mathbb{R}_{>0} \) for each \( v \in M_\mathbb{K} \) with \( S_v > 1 \) for a.e. \( v \in M_\mathbb{K} \) s.t. if we set

\[
W_K = \{ (x_v) \in M_\mathbb{K} : \| x \|_v \leq \delta_v \},
\]

then \( M_\mathbb{K} = K + W_K \).

Proof of Density Approximation: It is enough to prove the result for \( S = \delta_v \). A basis for the topology on \( M_\mathbb{K} \)

is sets of the form \( U = \prod_{v \neq v_0} U_v \), \( U_v \in K_v \) open and \( U_v = K_v \) for a.e. \( v \). Thus, we need to show that every such \( U \) contains an element of \( K \).

For such a \( U \) and let \( S' = \delta_v \in M_\mathbb{K} : U_v + (2\delta_v)^2 \).

Let \( x = (x_v) \in U \). Then exists an \( \varepsilon > 0 \) such that \( U \)

contains the set

\[
U_{x, \varepsilon} = \{ (y_v) \in M_\mathbb{K} : \| x - y \|_v < \varepsilon \text{ if } v \neq v_0, \| y \|_{v_0} < 1 \text{ if } v = v_0 \},
\]
The previous exercise yields $S_v \in H(\sigma_v)$ for each $v = M_k$ with
\[ S_v = 1 \quad \text{for a.e. } v \quad \text{and that} \quad |A_k| = K + W_k \quad \text{with} \]
\[ W_k = \sum_i (Y_i + A_{ik} : ||Y_i||_2 \leq \delta_v). \]

We now apply Corollary 1.5 to find a nonzero $\lambda_k$ so that
\[ ||\lambda_k|| < \frac{\epsilon}{\delta_v} \quad \text{for } v \neq \bar{v}, \]
\[ ||\lambda_k|| < \frac{1}{\delta_v} \quad \text{if } v = \bar{v}. \]

Let $z = \lambda_k x$ and write
\[ z = \alpha + w \in K + W_k. \]

Then we have
\[ x = \lambda \alpha + \lambda w \in K + \lambda W_k. \]

In particular, we have
\[ ||x - \lambda w||_v = ||\lambda||_v ||w||_v \left\{ \begin{array}{ll} < \epsilon & \text{if } v \neq \bar{v}, \\ = 1 & \text{if } v = \bar{v}. \end{array} \right. \]

Thus, $\lambda w + U_{x_k} \in U$ and we are done.

Section 1.2: Ideals:

In this section, we study the ideals more closely. We also begin to see how the ideals can be used to study...
the global field $K$. In particular, we give a very simple proof of the finiteness of the class number of $K$.

**Lemma 1.7:** The group $K^* = \mathbb{A}_k^*$ is discrete in $\mathbb{A}_K^*$.

**Proof:** Let $x \in \mathbb{A}_K^*$. Observe that

$$\mathbb{A}_K^* \longrightarrow \mathbb{A}_K^* \times \mathbb{A}_K^*$$

$$x \longmapsto (x, x^{-1}).$$

We have that

$$K^* \longrightarrow K \times K \leq \mathbb{A}_K^* \times \mathbb{A}_K^*$$

as well. We have already shown that $K \subset \mathbb{A}_K^*$ is discrete, so $K \times K \leq \mathbb{A}_K^* \times \mathbb{A}_K^*$ is discrete as well. Then

$$\mathbb{A}_K^* \cap K \times K = K^*$$

is discrete in $\mathbb{A}_K^*$. ■

Observe that the map

$$c : \mathbb{A}_K^* \longrightarrow \mathbb{R}_{>0},$$

$$x = (x, 1) \longmapsto \prod_{\mathfrak{p} \mid M_K} \|x\|_{\mathfrak{p}},$$

is a multiplicative homomorphism. The element $c(x) \in \mathbb{R}_{>0}$ is referred to as the content of the ideal $x$. 
Let \( J_K = \text{Ker}(c) = \{ x \in \mathbb{A}_K^* : c(x) = 1 \} \). Recall that last term we proved the product formula. Namely,

\[ \prod_{v \in M_K} \# M_v \cdot x_v = 1. \]

Thus, we have \( K^* \subseteq J_K \). Our next goal is to prove that \( J_K/K^* \) is compact. This fact will allow us to easily show that the class number of \( K \) is finite.

**Lemma 1.8:** Let \( K \) be a number field.

1. The map \( c \) is surjective.
2. The map \( c \) is continuous.

**Proof:** The first part is clear, as it follows from

\[ \mathbb{R} \subset C \subset \mathbb{R}_x, \] so one can get a surjection

by restricting to the archimedean places and having

\( c \) for all the nonarchimedean places.

We now prove 2. Let \( S \subseteq M_K \) be finite and set

\[ U_S = \prod_{v \in S} K_v^* \times \prod_{v \in S} \mathbb{Q}_v. \]

We have then the map \( x_1 \mapsto x_1 |_{U_S} \) is a continuous
from \( \mathbb{K}^x \) in \( \mathbb{R}^n_+ \) for each \( x \in \mathbb{K}_n \). The fact that the set

\[ S \]

is finite implies that

\[ C|_{U_s} : U_s \rightarrow \mathbb{R}^n_+ \]

is a continuous map (as in a finite product of continuous maps). However, we have \( \mathbb{K}^x \setminus \mathbb{K}_{x^s} = \bigcup_s U_s \) and so \( C \)

is continuous on \( \mathbb{K}^x \) as well.

This lemma shows that \( \mathbb{K} \setminus \mathbb{K}^x \) is closed.

**Prop. 1.9:** \( \circ \) \( \mathbb{K} \setminus \mathbb{K}^x \) is closed

\( \circ \) \( \mathbb{K} \) when considered as a subspace of \( \mathbb{K} \) has the

same topology as \( \mathbb{K} \) considered as a subspace

of \( \mathbb{K}^x \).

**Proof:** \( \circ \) Let \( x \in \mathbb{K} \setminus \mathbb{K}^x \). We need to find an open set \( U \subseteq \mathbb{K} \)

so that \( U \cap \mathbb{K}^x = \emptyset \) with \( x \in U \). We break into two cases.

**Case 1:** \( c(x) < 1 \).

This implies that there is a finite set \( S \subseteq \mathbb{K}_n \) as that

\[ \|x_v\|_p = 2 \quad \text{for} \quad v \in S \quad \text{and} \quad \prod_{v \in S} \|x_v\|_p < 1 \].

Let

\[ W_k = \left\{ (x')_{\mathbb{K}^x} : \|x_v\|_p < 2 \quad \forall \quad v \in S, \quad \|x_v\|_p \leq 1 \quad \forall v \in S \right\} . \]
This set is clearly open in $A_k$. We also clearly have $x \in W_k$.

Let $e$ be small, say $\varepsilon < \|x\|_1 \forall x \in S, \varepsilon > 0$. Then we have for all $W_k$,

$$\prod_{x \in W_k} \|x\|_1 < \prod_{x \in S} \|x\|_1 < \prod_{x \in S} \|x\|_1 < 1.$$ 

Thus, $\alpha \not\in J_k$ and so $W_k \cap J_k = \emptyset$.

\[ \text{Case 2: } c(x) > 1 \]

Let $e > c(x)$. Then exists a finite $S \subseteq M_k$ s.t.

1. $\forall x \in S$, then $\|x\|_1 = 1$

2. $\forall x \in S$, then $(\|x\|_1 < 1 \Rightarrow \|x\|_1 < \frac{1}{e} \Rightarrow e \subseteq K^x)$.

(Note this is possible since $\|x\|_1 = (\#L)^x$, i.e. $x$ is discrete !)

Let

$$W_k = \left\{ (x) \in A_k : \|x\|_1 - x, x \in c, \forall y \in S, \|y\|_1 < 1 \forall y \in S \right\}.$$ 

Choose $e$ small so that $\varepsilon < \|x\|_1 \forall x \in S$. Then for $x \in W_k$ we have

$$\prod_{x \in W_k} \|x\|_1 = \prod_{x \in S} \|x\|_1 \prod_{x \in S} \|x\|_1 \prod_{x \in S} \|x\|_1 \begin{cases} = e & \text{if } \|x\|_1 = 1 \forall y \in S \\ \leq \frac{1}{e} & \text{if } \|x\|_1 < 1 \forall y \in S \end{cases}.$$

\[ \text{Case 3: } c(x) < 1 \]

Note that the natural inclusion $A^x_k \to A_k$ is continuous.

Thus, for every open set $U \subseteq A_k$, we have that $U \cap J_k$ is open in the induced topology from $A_k^x$. Therefore, it is
enough to show that given any open set \( V \subseteq A^x_k, \ x \in J_k \), there exists an open set \( U \subseteq A^x_k \) so that \( \forall x \) and
\[
U \cup J_k \subseteq V \cap J_k.
\]

Let \( x \in J_k \). Let \( \epsilon > 0 \) and \( S \subseteq M_k \) finite.

\[
V_{\epsilon, S} \equiv \left\{ (x,y) \in A^x_k : \|x - y\|_1 < \epsilon, \ V \subseteq S \right\}, \|x\|_1 \leq 1 \ \forall x \in S.
\]

This set is clearly open in \( A^x_k \) and contains \( x \) if \( S \) contains all \( x \in M_k \) so \( \|x\|_1 < \frac{\epsilon}{3} \). If \( V \subseteq A^x_k \) contains \( x \), then \( \exists \epsilon, S \) such that \( V = V_{\epsilon, S} \).

1. \( S \) contains all \( x \in M_k \) so \( \|x\|_1 < \frac{\epsilon}{3} \).
2. \( \forall \epsilon < S \) and \( \|x\|_1 < \frac{\epsilon}{3} \) \( \exists x \in K_k \).
3. \( \exists \) an real number \( c(z) = \frac{\epsilon}{2} \forall z \in V_{\epsilon, S} \).

Then we have
\[
V_{\epsilon, S} \cap J_k = J_k \cap \left\{ (x,y) \in A^x_k : \|x - y\|_1 < \frac{\epsilon}{3}, \|x\|_1 \leq \frac{\epsilon}{3} \right\}.
\]

which is open in \( A^x_k \) (one can see this by looking at the radius \( a \leq \epsilon \)).

**Corollary 1.10:** \( A^x_k \) is compact.

**Proof:** Let \( x = (x) \in A^x_k \). Let \( \epsilon > 0 \) be chosen as in Lemma 1.5.

Set
\[
W = \left\{ (x', y) \in A^x_k : \|x' - y\|_1 < \|x\|_1, \|x\|_1 < \epsilon \right\}.
\]

This is a compact set. Let \( y = (y) \in J_k \). Observe that
Thus, applying Lemma 1.5 we see that there exists a nonzero \( p \in K^* \) such that

\[
\|p\|_v = \|y_v\|_v x, \|v_v\|
\]

and so \( p y_v \in W \). Thus,

\[
K^* \cdot (W \cap J_K) \subset J_K
\]

Since \( W \cap J_K \) is compact in \( J_K \) (we use the previous Proposition here), we have that \( J_K/K^* \) is the image of a compact set and thus is compact. 

We now give an elementary proof of the finiteness of the class number.

**Theorem:** Let \( K \) be a number field and \( C_K \) the class group of \( K \). Then \( \#C_K < \infty \).

**Proof:** Let \( I_K \) be the fractional ideals of \( K \) with the discrete topology. Let \( P_K \) be the principal fractional ideals of \( K \) so that \( C_K = I_K/P_K \). Define

\[
\pi : I_K \to I_K^{\prime}
\]

by

\[
x = (x_v) \mapsto \prod_{v \in \Omega} P_{v}(x_v)
\]

where \( \Omega = \{ \nu \} \) and \( P_{\nu} \) is a prime ideal of \( \nu \).

\[
K^* \cdot (W \cap J_K) \subset J_K
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\[
x = (x_v) \mapsto \prod_{v \in \Omega} P_{v}(x_v)
\]

where \( \Omega = \{ \nu \} \) and \( P_{\nu} \) is a prime ideal of \( \nu \).
This map is well defined because $x \neq 0_k$ for all but finitely many $v$. It is easy to see that this map is surjective. Let $\sigma I_k$ and $x \in k^x$ so that $\Phi(x) = \sigma I_k$.

Since $I_k$ has the discrete topology, to check that $\Phi$ is continuous it is enough to check that $\Phi^{-1}(\sigma I_k)$ is an open set in $k^x$. However,

$$\Phi^{-1}(\sigma I_k) = x \left( \prod_{v | \infty} k_v^x \times \prod_{v \nmid \infty} k_v^x \right),$$

which is clearly an open set in $k^x$. One checks that $\Phi(k^x) = \emptyset_k$, and so we have a surjection

$$J_k / k^x \twoheadrightarrow I_k / \emptyset_k.$$

However, a continuous surjection from a compact set onto a space with a discrete topology implies that $I_k / \emptyset_k$ must be finite, as claimed.

We will spend much more time relating ideals to statements about ideals, but from what we have done here...
We already see that
\[ \frac{\mathcal{O}_k^*}{K^* \prod_{v} k_v^* \prod_p \mathcal{O}_v^*} \xrightarrow{\sim} C_k \]

(\sigma)
\[ \to \prod_{v \in \mathcal{V}_k} k_v \cdot \mathcal{O}_v^* \] \in C_k.

Section 1.3 The Unit Theorem Revisited:

Let \( K \) be a number field, \( \mathcal{O}_k \) the ring of integers of \( K \), and we write \( U_k \) to denote \( \mathcal{O}_k^* \) when it is more convenient.

In introductory algebraic number theory one proves the

Unit Theorem.

**Theorem (Unit Theorem):** Let \( K \) be a number field, \( r, r_1, r_2 \) the numbers of real embeddings, and \( 2r_1, 2r_2 \) the numbers of complex embeddings. One has

\[ U_k \cong \mathbb{Z}^{r_1 + r_2 - 1} \times \mu(k)^r \]

where \( \mu(k) \) is the group of roots of unity contained in \( K \).

In this section we will use our adele machinery to give a
Rather simple proof of a generalization of this theorem.

Let \( S \subseteq \mathbb{M}_k \) be a finite set of planes, so that
\[ S_{\infty} = \{ v \mid v \in S \} \text{ is contained in } S. \]

**Def:** The \( S \)-units of \( K \) is the group
\[ U^K_S = \{ x \in K^\times : \|x\|_K = 1 \text{ for all } v \in S \}. \]

The \( S \)-integers of \( K \) is the ring
\[ O^K_S = \{ x \in K : \|x\|_K \leq 1 \text{ for all } v \in S \}. \]

Observe that if \( S = S_{\infty} \), then \( U^K_S = U_K \) and \( O^K_S = O_K \).

Moreover, for any finite set \( S \subseteq \mathbb{M}_k \) with \( S_{\infty} \subseteq S \) we have \( U_K \subseteq U^K_S \) and \( O_K \subseteq O^K_S \).

**Example:** Let \( K = \mathbb{Q} \) and \( S = S_{\infty}, p_1, \ldots, p_r \). Then
\[ U^K_S = \bigcap \{ p_1^{e_1} \cdots p_r^{e_r} : e_i \in \mathbb{Z} \}. \]
and
\[ O^K_S = \left\{ \frac{a}{b} : a, b = p_1^{e_1} \cdots p_r^{e_r}, e_i \in \mathbb{Z}, b \neq 0 \right\}. \]

We have the following generalization of the unit theorem,
Thm 4.11: Let \( K \) be a number field and \( S \subseteq M_K \) a finite set of places so that \( S_0 \subseteq S \). Then
\[
U_S^{S_0} = \mathbb{Z}^{#S - 1} \times \mathbb{R}^{#S}.
\]

Before we prove this theorem, we need the idele analog to the classical result that for any constant \( c \) we have
\[
\sum_{\mathfrak{q} \subseteq I_K \mid \mathfrak{a}_1} \psi(\mathfrak{q}) \cdot \psi(\mathfrak{a}_1) < \infty.
\]

Lemma 4.12: Let \( c, c_0, c_1 \) be constants so that
\[0 < c_0 < c_1 < \infty.\]
Then, we have
\[
\sum_{\mathfrak{q} \subseteq I_K \mid \mathfrak{a}_1} \psi(\mathfrak{q}) \cdot \psi(\mathfrak{a}_1) < \infty.
\]

Proof: Let
\[
W = \sum_{\mathfrak{q} \subseteq I_K \mid \mathfrak{a}_1} \chi(\mathfrak{q}) \cdot \psi(\mathfrak{a}_1), \quad \mathfrak{a}_1 = (a_1)_{\mathfrak{q} \subseteq I_K \mid \mathfrak{a}_1}.
\]
We have that \( W \) is compact because \( W \) is inner-compact to \( \prod \mathcal{O}_K^\times \) compact set,
\[
\prod \mathcal{O}_K^\times \text{ compact set},
\]
which is compact. Thus, \( W \mathcal{O}_K^\times \) is compact and since \( K^\times \subseteq \mathcal{O}_K^\times \) is discrete, we have \( W \mathcal{O}_K^\times \) must be finite.
Prop. (Thm 1.11): Let $\mathbb{A}_k^x$ be the set

$$\bigcap_{\lambda \in \mathbb{A}_k^x} \{ x \in \mathbb{A}_k^x : \forall \lambda \in \mathbb{A}_k^x \; \exists \lambda \in \mathbb{A}_k^x \}. $$

This is clearly an open set in $\mathbb{A}_k^x$. Let $J_s = J_k \cap \mathbb{A}_k^x$

where we recall that $J_k = \ker(C)$, where $C(x) = \prod_{x \in \mathbb{A}_k^x}$.

It is not difficult to see that $U_k^s = K^x \cap J_s$ and so

$$J_s/U_k^s = J_s/K^x \cap J_s \to J_k/K^x.$$  

One has that $J_s/U_k^s$ is an open subgroup of $J_k/K^x$

with continuous index, and $U_k^s$ is also a closed subgroup.

(This is a standard result about topological groups).

The fact that $J_k/K^x$ is compact and $J_s/U_k^s$ is a

closed subgroup of $J_k/K^x$ implies that $J_s/U_k^s$ is also

compact.

Suppose first that $S = S_{\infty}$. Consider the map

$$\lambda: \mathbb{A}_k^x \to \mathbb{R}_+ \oplus \cdots \oplus \mathbb{R}_+ \text{ \# S copies}$$

$$(x_1) \mapsto (\log ||x_1||_v)_{v \in S}.$$  

This map is surjective and continuous (we can adjust

the $x_1$ for all $v \in S$ to make it so.) Observe that

$$\lambda(J_s) = \left\{ (x_1, \ldots, x_S) : \sum x_i = 0 \right\}.$$  

Let $H = \lambda(J_s)$. We have $U_k^S \subseteq J_s$, and so
\[ \Lambda = \lambda(U_k^S) \subseteq H \subseteq \mathbb{R}_{2_0}^{#S} \]. We have that \( \Lambda \) is

direct by Lemma 1.12. The dimension of \( H \) is

\( #S - 1 \). We next want to show \( \Lambda \) spans \( H \).

Observe that

\[ \frac{\lambda(T_{s})}{\lambda(U_k^S)} = \frac{H}{\Lambda} \]

is the continuous image of \( T_{s}/U_k^S \), a compact set. Thus,

\( H/\Lambda \) is a compact set. This gives that \( \Lambda \) spans

\( H \) because \( H/\Lambda \) is a finite or an infinite

sum of \( R^2 \). If it were a finite sum of \( R^2 \)'s the would

violate the compactness. Thus we have that \( \Lambda \) is

a lattice and so \( \Lambda \cong \mathbb{Z}^{#S-1} \cong \lambda(U_k^S) \). We

see that \( \text{Ker} \lambda \cap U_k^S = \{ x \in \mathbb{Z}^2 : \|x\|_1 = 1 \} + N_k \).

This is a finite group with \( \text{Ker} \lambda \cap U_k^S = \mu(\{k\}) \). Thus,

\[ U_k^S \cong \mathbb{Z}^{#S-1} \times \mu(\{k\}) \).

Suppose now that \( S = S_0 \cup S_1 \) for \( S \) some finite

set of finite places. Consider the map

\[ \lambda_{S} : \Lambda_{k,S} \longrightarrow \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \]

\[ \#S \].
\[(x_0) \mapsto (\sigma(x_0), x_0) \nu_s s_f.\]

This is a surjective map and continuous when we give \(\mathbb{Z}^{\#s_f} \nu_s s_f\) the discrete topology. Observe that

\[\text{Ker } (\lambda_s) = \Lambda^s_{K, s_f}.\]

This is a surjective map even when we restrict it to \(J_s\).

Since \(J_s \nu_s s_f\) is compact, we have \(\mathbb{Z}^{\#s_f} / \Lambda^s_{K, s_f} \subset \text{Ker } (\lambda_s)\).

\[x_0 \in \Lambda^s_{K, s_f} \quad \text{if and only if} \quad \lambda_s^n(x_0) = 1 \quad \text{for all} \quad n \in \mathbb{Z}^{\#s_f} \nu_s s_f.\]

Since \(J_s \nu_s s_f\) is compact, we have \(\text{Ker } (\lambda_s) \simeq \mathbb{Z}^{\#s_f} \nu_s s_f\).

\[\text{Ker } (\lambda_s|_{U^s_K}) = U^s_K \cap \text{Ker } (\lambda_s) = U^s_K \cap \Lambda^s_{K, s_f}.\]

\[U^s_K \subset \Lambda^s_{K, s_f} \quad \text{if and only if} \quad \lambda_s^n(x_0) = 1 \quad \text{for all} \quad n \in \mathbb{Z}^{\#s_f} \nu_s s_f.\]

When the last equality follows from the case when \(s = s_f\).
In this chapter we give the statements of the main
theorems of global class field theory. We will use these
results in subsequent chapters to study various applications.
If there is time we will return to give outlines of the
proposals later in the course.

Section 2.1 In terms of ideles:

Recall that in the last chapter we showed that
the group \( \mathbb{A}_K^\times / K^\times \) surjects onto the ideal class group
\( C_k \). We call \( \mathbb{A}_K^\times / K^\times \) the idele class group and
denote it by \( C_k \). (This should cause no confusion with
the notation \( C \) for the complex numbers or on idele class
groups will always have a subscript, though context should
also make clear which one we are referring to.) Before we can
state the results we need a little more background.
Let $L/K$ be a finite abelian extension of number fields. Let $v \in M_K$ and $w \in M_K$ so that $w/v$. Recall the decomposition group $D(w)$ is defined by

$$D(w) = \left\{ \sigma \in \text{Gal}(L/K) : \sigma(w) = w \right\}.$$ 

It is easy to see that $D(w) \cong \text{Gal}(L_w/K_v)$. Thus, from local class field theory we have the local Artin map

$$\Phi_v : K_v^* \rightarrow D(w) \subset \text{Gal}(L/K).$$

**Lemma 21**: The subgroup $D(w)$ of $\text{Gal}(L/K)$ and the map $\Phi_v$ into $D(w)$ are independent of the choice of $w$ over $v$.

**Proof**: Let $w'$ be another place dividing $v$. We can write

$$w' = \sigma w$$

for some $\sigma \in \text{Gal}(L/K)$. This $\sigma$ extends to a homomorphism $\sigma : L_w \rightarrow L_{w'}$ so that $K_v$ is fixed by $\sigma$. It is clear that

$$D(w') = \sigma D(w) \sigma^{-1}.$$ 

Since $\text{Gal}(L/K)$ is assumed to be abelian, we have
\[ D(w) = D(w'), \]

as claimed.

We now need to discuss norms of ideals. Recall the isomorphism

\[ L \otimes_K K_v \cong \prod_{w \mid v} L_w. \]

From this we see that for \( x \in L \),

\[ N_{L/K} x = \prod_{w \mid v} N_{L_w/K_w} x. \]

Where the equality is in \( K_v \). Let \( x = (x_w) \in A_K^\times \). We define the norm of \( x \) down to \( A_K^\times \) by

\[ N_{K/K_v}(x) = \left( \prod_{w \mid v} N_{L_w/K_w}(x_w) \right). \]

**Exercise:** Prove the following diagram commutes:

\[
\begin{array}{ccc}
L^\times & \rightarrow & A_L^\times \\
\downarrow N_{K/K_v} & & \downarrow \text{tr} \\
K^\times & \rightarrow & A_K^\times \\
\end{array}
\]

In particular, we get a commutative diagram:

\[
\begin{array}{ccc}
C_L & \rightarrow & C_L \\
\downarrow N_{K/K_v} & & \downarrow \text{tr} \\
C_K & \rightarrow & C_K \\
\end{array}
\]
Though it will not be needed until we come back to ideals, we also briefly recall some of the main properties of the Frobenius element. Let \( K \) be a finite Galois extension of a number field with \( G = \text{Gal}(K/K_0) \). Let \( \omega \in M_L \) and \( \omega, M_L \in \omega \). Assume that \( \omega \) is unramified over \( K \). We recall that in this case we have an isomorphism,

\[
\text{Gal}(K_{\omega}/K_0) \cong \text{Gal}(K_{\omega}/K).
\]

The group \( \text{Gal}(K_{\omega}) \) is cyclic with canonical generator \( \text{Frob}_{K_{\omega}/K_0} \). This Frobenius element, i.e., the map \( x \rightarrow x^\omega \) where \( \omega = \#K_0 \).

Thus, \( \text{Gal}(K_{\omega}/K_0) \) and hence \( D(\omega) \) must be cyclic. We call the generator of \( \text{Gal}(K_{\omega}/K_0) \cong D(\omega) \) the Frobenius element as well and denote it by \( \text{Frob}_\omega \). Many authors use the notation \( (\omega, K_{\omega}/K) \) to denote the Frobenius element in \( \text{Gal}(K_{\omega}/K) \). Observe that we have:

1. \( \text{Frob}_\omega \omega = \omega \);
2. \( \forall \alpha \in K \), \( \text{Frob}_\omega \alpha = \alpha^\omega \mod \omega \).

Lemma 3.2: Let \( \omega = \sigma \omega \in M_L \) be a second place dividing \( \nu \).

Then, \( D(\omega) = \sigma D(\omega) \sigma^{-1} \) and \( \text{Frob}_\omega = \sigma \text{Frob}_\omega \sigma^{-1} \).
Let $x \in D(w)$. Then we have
\[
\begin{align*}
\sigma \tau \sigma^{-1}(w) &= \sigma \tau \sigma^{-1}(w) \\
&= \sigma \tau w \\
&= \sigma \omega w \\
&= w'.
\end{align*}
\]
Thus, $D(w') \subseteq D(w) \sigma^{-1}$. These groups have the same order, thus they must be equal.

Let $x \in D(w)$. Then we have
\[
\sigma \text{Frob}_w \sigma^{-1}(x) = \sigma \text{Frob}_w (\sigma^{-1}(x) + y)
\]
for some $y \in W$.

Hence, $(\sigma^{-1}(x))^2 = \sigma^{-1}(x) + y$ and so we have $\sigma \text{Frob}_w \sigma^{-1}(x) \in x^2 (\text{Frob}_w \omega w)$, and the uniqueness of Frobenius gives the result. \(\square\)

We know from algebraic number theory that $G$ acts transitively on the set $\{x \in M_K : \text{val}_v \frac{1}{2}\}$. Thus, we have that
\[
\sum_{x \in \mathcal{E}_u} \text{val}_v \frac{1}{2} \leq \text{Gal}(L/K)
\]
is a conjugacy class by the previous lemma. We write $F_v/K$ to denote this conjugacy class. Let $S \subseteq M_K$ be a finite set of places containing those that ramify in $L$ as well as the ones dividing $v$. Then $F_v/K$ is a map from $M_K \setminus S$ to the conjugacy class of $G$.

Note that if $G$ is abelian, then $F_v/K(v)$ is a single element, which
we consider as an element of $\text{Gal}(E/K)$ rather than a conjugacy class.

\[ \text{Lemma 2.3: Consider a tower of number fields:} \]

\[ K \subset L \subset E \]

with $u \in M_K$, $v \in M_L$, $w \in M_E$ s.t. $u | v | w$. Assume $w$ is unramified over $u$. Then we have

\[ (w, E/L) = (w, E/K)^f(v/w) \]

\[ \text{Proof: Let } h_w \in \ell_f \in h_w \text{ be the inert or ramified field. By definition we have} \]

\[ f(v/w) = [h_v : h_w] \]

and

\[ (w, E/L) \]

is the # $h_v$ - power map. However, $(w, E/K)$

is the # $h_v$ - power map. And so

\[ (w, E/L)^f(v/w) \]

is the # $h_w \cdot [h_v : h_w] = # h_v$ - power map, i.e.

\[ (w, E/K)^f(v/w) = (w, E/L) \]

as claimed. \[ \Box \]
Lemma 2.4: Let \( L_1 \) and \( L_2 \) be \( \ell \)-adic extensions of \( K \). Let \( \omega \in L_1 \) and \( \nu \in L_2 \). Then we have

\[
\omega, L_1, L_2, \nu, \gamma_1, \gamma_2, \sigma_1, \sigma_2,
\]

under the natural injection

\[
\text{Gal} \left( \frac{L_1 \nu}{K} \right) \to \text{Gal} \left( \frac{L_2 \nu}{K} \right) \times \text{Gal} \left( \frac{L_1 \omega}{K} \right).
\]

\( \sigma \mapsto (\sigma_1, \sigma_2). \)

Proof: Exercise.

We now come to the statement of the main theorem in terms of ideals. We begin with the following Proposition:

Prop 3.5: There is a unique continuous homomorphism

\[
\Phi_v : \mathbb{A}_K^\times \to \text{Gal} \left( \frac{K^{ab} \nu}{K} \right)
\]

satisfying for any finite abelian extension \( L \subset K \)

and \( w, v \in M_v \), \( u \in M_u \), the following diagram commutes:

\[
\begin{array}{ccc}
K_v & \xrightarrow{\Phi_v} & \text{Gal} \left( \frac{L^{ab} \nu}{K_v} \right) \\
\downarrow & & \downarrow \\
\mathbb{A}_K^\times & \xrightarrow{x \mapsto \Phi_v(x, u)} & \text{Gal} \left( \frac{L^{ab} \nu}{K_v} \right).
\end{array}
\]
We need to check that this map makes sense and satisfies the required properties. We know that for all but finitely many \( v \in M_K, \mathbb{L}/K_v \) is unramified. We also know that \( x_v \in C^\times_v \) for all but finitely many \( v \). Recall from CRT that if \( L/K_v \) is unramified and \( x_v \in C^\times_v \), then \( \Phi_v(x_v) = 1 \). Thus, \( \Phi_v(x_v) = 1 \) for all but finitely many \( v \) and so the map makes sense when we consider \( \Phi_v(x_v) \in \text{Gal}(L_w/K_v) \). Let \( K \leq L \leq E \) be a finite abelian extension.

The local reciprocity theorem gives that \( \Phi_v|_L(x) = \Phi_v|_K(x) \) and so there exists a 1 homomorphism \( \Phi_v: M_v \rightarrow \text{Gal}(K_v^\times /K_v) \) such that \( \Phi_v|_L(x) = \Phi_v|_K(x) \) for all \( v \) in a finite set of places.

Let \( S \subseteq M_K \) be a finite set of places. Using what we know of the local action map we obtain that the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{A}^\times_K & \xrightarrow{\Phi_v|_L} & \text{Gal}(E_L) \\
\downarrow N_{L/K} & & \downarrow N_{L/K} \\
\mathbb{A}^\times_K & \xrightarrow{\Phi_v|_K} & \text{Gal}(E_K) \\
\end{array}
\]

If we set \( L = E \) we obtain that \( N_{L/K}(\mathbb{A}_K^\times) \) is contained in the kernel of \( \Phi_v|_K \). We will see that when \( \Phi_v|_K \) is continuous.
Theorem 2.6 (Global Reciprocity Law): Let $K$ be a number field. The map

$$
\phi_K : \mathbb{A}_K^x \to \text{Gal}(K^{ab}/K)
$$

(global action map) has the following properties:

1. $\phi_K(\mathbb{A}_K^x) = 1$;

2. for every $L/K$ finite abelian, the group is cyclic:

$$\phi_{L/K} : \mathbb{A}_K^x \cap \text{Nm}(\mathbb{A}_L^x) \to \text{Gal}(L/K).$$

A better way to write point 2 is to say that $\phi$ defines an immersion:

$$\phi_{L/K} : \mathbb{A}_K^x \cap \text{Nm}(\mathbb{A}_L^x) \to \text{Gal}(L/K).$$

Thm 3.7 (Existence Theorem): Let $K$ be an algebraic closure of $k$. For every open subgroup $N \subset \mathbb{C}_k$ of finite index $\exists$ a $k$-algebraic extension $L/K$ with $L \supset K$ s.t. $\text{Nm}_{L/K} \mathbb{C}_K = N$.

Recall that for number fields all subgroups of finite index in $\mathbb{C}_K$ are open!
The map \( L \to \text{Nm}(C_L) \) defines a bijection between the set of finite abelian extensions of \( K \) to the set of open subgroups of finite index in \( C_K \). Moreover, we have

\[
L_1 \subseteq L_2 \iff \text{Nm}(C_{L_1}) \subseteq \text{Nm}(C_{L_2})
\]

\[
\text{Nm}(C_{L_1}, C_{L_2}) = \text{Nm}(C_{L_1}) \cap \text{Nm}(C_{L_2})
\]

\[
\text{Nm}(C_{L_1}, n_{L_2}) = \text{Nm}(C_{L_1}) \text{Nm}(C_{L_2})
\]

Note that we have

\[
\varphi_K : \mathbb{M}_{K^*}^\times \to \text{Gal}(\bar{K}/K) \cong \mathbb{G}_K^\text{ab}
\]

is a continuous map, and since we got an isomorphism at all finite \( L \) in, we have that the image of \( \varphi_K \) is dense in \( \mathbb{G}_K^\text{ab} \).

Let \( v \in \text{val}(K) \) and write \( (K_v^*)^0 \) for the connected component of \( K_v^* \) that contains the identity. For example, if \( K_v^* = \mathbb{R}^* \), then \( (K_v^*)^0 = \mathbb{R}_{>0} \).

The fact that \( \varphi_K \) is continuous, gives that \( \varphi_K \) must kill \( (K_v^*)^0 \) for each \( v \in \text{val}(K) \), i.e., it must kill \( \prod_{v \in \text{val}(K)} (K_v^*)^0 \). The fact that the second is closed implies that \( \varphi_K \) must kill \( \prod_{v \in \text{val}(K)} (K_v^*)^0 \).

The set \( \prod_{v \in \text{val}(K)} (K_v^*)^0 \) is the connected component of the identity, which will follow from a lemma to be stated in a moment. Let

\[
D_K = \prod_{v \in \text{val}(K)} (K_v^*)^0
\]
Lemma 2.9: Let $G$ be a locally compact topological group.

The connected component of the unit element of $G$ is equal to the intersection of the open subgroups of $G$.

If $N$ is a normal closed subgroup of $G$, the connected component of the unit element of $G/N$ is equal to the closure of the image in $G/N$ of the connected component of the unit element of $G$.

This lemma is a fairly standard result of topological groups, it can be found for example in Bourbaki.

\[ \text{Exercise: Show that } U_0 = \bigcap \{ C^+_x \}_{x \in X} \text{ is the connected component of the identity in } \mathbb{A}_k. \]

\[ \text{Lemma 2.9} \]

We can use the previous exercise to conclude that $D_k$ is the connected component of $1$ in $C_k$. Moreover, we can also use Lemma 2.9 to conclude the following:

\[ \lim_{x \to 0^+} C^+_x/N = C^+_x/D_N \cong C^+_x/D_k \]

\[ \cong \mathcal{O}_k \]

\[ \lim_{x \to 0^+} \varpi_{C^+_x} \cong C^+_{\mathcal{O}_k}. \]

Thus, $C^+_x/D_k \cong C^+_{\mathcal{O}_k}$. 

One should be careful here in that we are relying on our assumption that \( K \) is a number field! In the function field case we have that \( G_k \) is injective not surjective. The map \( \phi_k \) gives an isomorphism of \( G_k \) onto \( W_k \), the Weil group, but it is defined analogously to the local Weil group, i.e., it is dense in \( G_{\mathbb{A}}^{ab} \).

Note that even though we have this description of \( G_{\mathbb{A}}^{ab} \)
\[
\mathbb{C}^*/D_k,
\]
the group \( \mathbb{C}^*/D_k \) is in general very complicated so this is not really very helpful!

Before we give the adele theoretic formulation of global class field theory, we give a couple of applications of what we have done. While the first does not require CFT, we will need it when we compute examples later on and is important in its own right, especially in its generalization to other reductive algebraic groups (this is the \( \ell \)-adic case!). This is also known as the strong approximation theorem for \( \mathbb{Q} \).

**Theorem 2.10:** We have
\[
\mathbb{A}_\mathbb{Q}^{\times} = \mathbb{Q}^\times \left( \mathbb{R}_+ \times \prod_{\ell} \mathbb{Z}_\ell' \right).
\]
**Proof:** Let \( x = (x_0, x_2, y_3, \ldots) \in \mathbb{A}^\mathbb{R}^\mathbb{Q} \). We want to show we can write \( x = t(\beta_0, \beta_2, \gamma_3, \ldots) \) with \( t \in \mathbb{A}^\mathbb{Q} \), \( \beta_0 \in \mathbb{R}^\mathbb{Q} \), and \( \gamma_p \in \mathbb{Z}^\mathbb{Q} \) for all primes \( p \). Let \( S = M\mathbb{A}^\mathbb{Q} \) be the set of finite sets of monomial places with \( x_p \in \mathbb{Z}^\mathbb{Q} \). Let \( t = \lim_{p \to \infty} (x_0) \prod_{p \in S} p_{p}\text{gcd}(x_p) \) and set \( \beta_0 = \frac{x_0}{t} \), \( \gamma_p = \frac{x_p}{t} \). Note that \( \gamma_p > 0 \) and \( |\gamma_p| = 1 \) for all \( p \).

(Essentially we have factored out the finitely many \( p \)'s that occur in the \( x_p \)'s.) This gives our decomposition. \( \square \)

This result is true in more generality, the essential property of \( \mathbb{A} \) is that it has class number 1. We will return to prove this in the more general case after we have discussed global class fields.

As it may be helpful, we briefly look at what we have for extensions of \( \mathbb{A} \) in terms of GCFT and its relation to GCFT. For example, let \( K \) be a field we keep track of the Frobenius element. One should really work out all the details in what the term says explicitly for cyclotomic extensions of \( \mathbb{A} \).

Let \( K \) be an unramified extension of \( \mathbb{A} \). Then we have \( K_v / \mathbb{A}_v \) is an unramified extension for \( v \mid P \). GCFT says the following diagram commutes:
We have already seen that \( \text{Gal}(K/\mathbb{Q}_p) = D(V) \), which tells us how we see the decomposition group. We saw in \( \text{ECFT} \) that the inertia group corresponds under the local norm map to \( \mathbb{Z}_p^\times \). This shows how the inertia appears in \( \text{ECFT} \) as

\[
/A_{\mathbb{Q}}^\times /Q^\times = \prod_p \mathbb{Z}_p^\times.
\]

Finally, recall that under \( \phi_p \) we had \( p \in \mathbb{Q}_p^\times \) mapping to \( Frob_p \in \text{Gal}(K/\mathbb{Q}_p) \). We know that \( p \in \mathbb{Q}_p^\times \) maps to \((1,1,\ldots,1,p,\ldots) \in A_{\mathbb{Q}}^\times \), and to the element \( p\)-place \((1,1,\ldots,1,p,\ldots) \) maps to \( (p,K/\mathbb{Q}) \) in \( \text{Gal}(K/\mathbb{Q}) \).

We end this section by giving a simple proof of the law of quadratic reciprocity. We will possibly come back to more general reciprocity laws later on, but since this familiar in any case. Consider the field \( \mathbb{Q} \sqrt{-\frac{p}{4}} \) for an odd prime \( p \), and the quadratic field \( K = \mathbb{Q} \sqrt{-\frac{p}{4}} \). Note that the discriminant of \( K/\mathbb{Q} \) is \( p \) and so \( K \) is ramified only at \( p \). We use the isomorphism \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2 \mathbb{Z} \).
Throughout, for any $x \in \mathbb{Q}^*$, we can use the fact that $\phi_2$ is a homomorphism to conclude that in order to compute $\phi_2(x)$, it is enough to be able to compute $\phi_2(-1)$ and $\phi_2(2)$ for all primes $q$.

We know that for any $x \in \mathbb{Q}^*$, $q \neq p$, we have

$$\phi_2(x) = \left( \frac{1}{q^*} \right) \phi_2(x)$$

Thus, for any given $x$ we know that $\phi_2(x) = 1$ for almost every $q$ (mod 2, except those where ord$_q(x) + 1 = 0$ or $p$). This is essentially what we said when defining the global automorphisms.

Observe that for $v = \infty$, we have $K_v/\mathbb{Q}_v = \mathbb{Q}/\mathbb{Q}$ if $(-1)^{p_2^v} = 1$ and $K_v/\mathbb{Q}_v = \mathbb{Q}^*/\mathbb{Q}$ if $(-1)^{p_2^v} = -1$. Recall from CRT that if $K_v/\mathbb{Q}_v = \mathbb{Q}/\mathbb{Q}$, then $\phi_v$ is trivial and so $\phi_v(-1) = 1$, then $\phi_v(x) = 1 = (-1)^{p_2^v}$. If $K_v/\mathbb{Q}_v = \mathbb{Q}^*/\mathbb{Q}$, then $\phi_v(-1) = -1 = (-1)^{p_2^v}$. Thus, for $v = \infty$, we have $\phi_v(-1) = (-1)^{p_2^v}$. We know that $\phi_2(-1) = 1$ for all $q \neq p$.

So the product formula gives $\phi_2(-1) = (-1)^{p_2^v}$ as well.

It is also clear that $\phi_2(q^*_q) = 1$ for all primes $q_1, q_2$ except possibly if $q_1 = q_2 \neq p$. First suppose that $q_1 = p$. In this case the only possible non-trivial term is when $q_1 = p$. However, we are then computing $\phi_2(p)$, which must be 1 as well. This follows from the fact that $\mathbb{Q}^*/\mathbb{Q}$ is trivial and so $1 = \phi_2(p) = \bigcap \phi_2(p)$.

Suppose now that $q_1 \neq p$. Then we have 2 cases, when we may have nontrivial $\phi_2(q_2)$, $q_1 = q_2 = p$. Suppose first
that \( q_1 = p \). Note that in this case we have \( q_2 \in \mathbb{Z}_p^\times \), we have that the extension \( K_2/Q_2 \) is ramified. We know that 
\[ \Phi_2 \left( q_2 \right) = 1 \quad \text{iff} \quad q_2 \text{ is a norm of } K_2 \text{ i.e. } q_2 \in \text{Norm}_{K_2/Q_2}(K_2^\times). \]

The fact that the extension is totally ramified gives that 
\[ [ \mathbb{Z}_p^2 : \mathbb{Z}_p^2 \cap \text{Norm}_{K_2/Q_2}^\times ] = 2. \]

However, we can also see that the only subgroup of \( \mathbb{Z}_p^2 \) is \( (\mathbb{Z}_p^2)^\times \times (1 + p\mathbb{Z}_p)^\times \) of index 2 is 
\[ ((\mathbb{Z}_p^2)^\times)^2 \times (1 + p\mathbb{Z}_p)^2 = \left( (\mathbb{Z}_p^2)^\times \right)^2 \times (1 + p\mathbb{Z}_p) = (\mathbb{Z}_p^2)^\times. \]

Thus, 
\[ q_2 \text{ is a norm of } \mathbb{Z}_p^\times \text{ iff } q_2 \in (\mathbb{Z}_p^2)^\times, \quad \text{i.e. } \quad q_2 \in (\mathbb{Z}_p^2)^\times. \]

by Hensel's lemma. Thus, we have 
\[ \Phi_2 \left( q_2 \right) = \left( \frac{q_2}{p} \right). \]

Suppose now that \( q_1 = q_2 \). Write \( q = q_1 = q_2 \) in some notation. We wish to compute \( \Phi_2(q) \). We know that \( K_2/Q_2 \) is unramified and so \( \Phi_2(q) = \text{Frob}_2 \). The fact that \( K/Q \) is unramified at \( q \) implies that \( q_1^{\sigma_2} = q_2^{\tau_2} \) or \( q_1^{\sigma_2} = q_2^{\tau_2} \). We have that the decomposition group is generated by the Frobenius map since the inertia group is trivial is of size \( f = f(\mathbb{Q}_p) \) or \( f(\mathbb{F}_p/q) \) (depending on the factorization of \( q \) in \( K \)). However, this shows the Frobenius is trivial if \( q_1 \) splits as in the cases of split, \( e = 2 \) and 
\[ f = 1 \text{ and so } \text{Frob} = 1. \] Similarly for the mentioned cases.

However, we know that \( q_1 \) splits in \( K \) precisely when 
\[ K_2 = \mathbb{Q}_2 \quad (\text{since } f=1), \text{ this must be the same field because} \]
\[ [K_2:Q_2] = f(e) \quad \text{Thus, } \Phi_2(q_2) = 1 \quad \text{iff} \quad \mathbb{Q}_2 \left( \sqrt{(-1)^{f(e)}p} \right) = \mathbb{Q}_2, \]
i.e. \( \sqrt{(-1)^{f(e)}p} \in \mathbb{Q}_2 \), i.e. \( (-1)^{f(e)}p \in \mathbb{Q}_2 \). This gives (at least if \( q_2 \equiv 0 \pmod{p^2} \)), 
\[ \Phi_2(q_2) = \left( \frac{(-1)^{f(e)}p}{q_2} \right). \]
Thus, we can apply the fact that \( \mathbf{q} \in \ker(\phi_2) \) to obtain

\[
1 = \phi_2(q) = \prod_v \phi_v(q)
\]

\[
= \phi_\mathbb{Q}(q) \phi_p(q).
\]

We then have (for \( q \neq p, \mathbb{Q} \))

\[
1 = \left( \frac{a}{p} \right) \left( \frac{-1}{q} \right) \left( \frac{p}{q} \right)
\]

\[
= \left( \frac{a}{p} \right) \left( \frac{-1}{q} \right) \left( \frac{p}{q} \right)
\]

\[
= \left( \frac{a}{p} \right) \left( \frac{p}{q} \right) \cdot \left( \frac{-1}{q} \right)^{p-1}
\]

\[
= \left( \frac{a}{p} \right) \left( \frac{p}{q} \right) \cdot (-1)^{\frac{p-1}{2}}
\]

\[
\Rightarrow \left( \frac{a}{p} \right) \left( \frac{p}{q} \right) = (-1)^{\frac{p-1}{2}} (-1)^{\frac{q-1}{2}}
\]

\[\text{for } p, q \text{ odd primes.}\]

\[\frac{2}{p}\]

**Exercise:** Work out \( \left( \frac{2}{p} \right) \) following these lines and computations.

\[\text{mod } 8\]
In this section we will give the definitions of Ray class groups and see how they really encompass all the extensions in that each is contained in a Ray class field. We will also set up some of the necessary notation for the next section about GCFT in terms of ideals.

Let $K$ be a number field, $I_K$ the fractional ideals of $K$, and $S \subseteq M_K$ a finite set of places of $K$. We let $I_K^S$ denote the subgroup of $I_K$ that is generated by the finite places not in $S$, i.e., the free abelian group generated by prime ideals in $O_K$ not corresponding to places in $S$. Let

$$K^S = \left\{ a \in K^* \mid \text{ord}_v(a) = 0 \quad \forall v \in S, \forall \infty \right\}.$$ 

We have a natural embedding

$$\iota : K^S \hookrightarrow I_K^S$$

$$\alpha \mapsto (\alpha).$$

We identify $K^S$ with its image in $I_K^S$. Write $S \subseteq \mathcal{M}_K$ to denote the set of finite places.

**Lemma 2.11:** Let $S \subseteq S_f$ be a finite set. The following sequence is exact:

$$0 \rightarrow \mathcal{O}_K^* 
\rightarrow K^S 
\rightarrow I_K^S 
\rightarrow C_K 
\rightarrow 0.$$
Proof: We begin by choosing $I_K^5 \to C_k$. Let $\mathfrak{o}_I^{\prime} \subseteq \mathfrak{I}_k$

frequently ideal in $\mathcal{I}_K$ and write $[\mathfrak{o}_1]$ for the class of an $\mathfrak{I}_k$ in $C_k$. Write $\mathfrak{o}_I = \mathfrak{o}_1 \mathfrak{I}_K^{-1}$ for $\mathfrak{I}_K$ and $\mathfrak{I}_K^{-1}$ integral ideals. Given any nonzero element $\mathfrak{c} \in \mathfrak{C}_k$, we know $\mathfrak{c} | (\mathfrak{c})$ and so $\mathfrak{o}_I | (\mathfrak{c}) = \mathfrak{I}_K^{-1} (\mathfrak{c})$ is an integral ideal. This shows that we can choose an integral ideal to represent $[\mathfrak{o}_I]$, namely $[\mathfrak{o}_I] = (\mathfrak{c})$. Without loss of generality we may assume $\mathfrak{o}_I$ is an integral ideal. Write

$$\mathfrak{o}_I = \prod_{\mathfrak{p} \in \mathfrak{P}_k} \mathfrak{p}^{r_{\mathfrak{p}}(\mathfrak{o}_I)} \mathfrak{P}$$

where $\mathfrak{P} \in \mathcal{I}_K$. For each $\mathfrak{p} \in \mathfrak{P}_k$, choose $\mathfrak{te} \in \mathfrak{p} \setminus \mathfrak{p}^2$, i.e., an element so that $\operatorname{ord}_{\mathfrak{p}}(te) = 1$. The Chinese Remainder Theorem gives $x \in C_k$ s.t.

$$x \equiv t_{\mathfrak{p}}^{r_{\mathfrak{p}}(\mathfrak{o}_I)} \mod \mathfrak{p}^{r_{\mathfrak{p}}(\mathfrak{o}_I)+1}$$

for all $\mathfrak{p} \in \mathfrak{P}_k$. This gives $x \in C_k$ s.t. $\operatorname{ord}_{\mathfrak{p}}(x) = r_{\mathfrak{p}}(\mathfrak{o}_I)$

for each $\mathfrak{p} \in \mathfrak{P}_k$. Thus, $(x) = \prod_{\mathfrak{p} \in \mathfrak{P}_k} \mathfrak{p}^{r_{\mathfrak{p}}(\mathfrak{o}_I)} \mathfrak{P}$ with $\mathfrak{P} \in \mathcal{I}_K$. Then, we have that $I_K^5 \to C_k$.

Thus, $\mathfrak{c}$ is a $\mathfrak{P} \in \mathcal{I}_K$. The rest is clear. If $\mathfrak{o}_I \subseteq I_k$ maps to $0$ in $C_k$, we must have $\mathfrak{o}_I \subseteq I_k \cap K^\times = K^\times$. The element $\mathfrak{o}_I = (\mathfrak{c}) \in K^\times$ is uniquely determined up to a unit, i.e.,

$$0 \to C_k^\times \to K^\times \to I_k^\times$$
is exact. This combined with the fact that \( \pi_{C_k} \rightarrow C_k \)
with kernel \( K^S \) gives the result.

**Def:** A modulus for the field \( K \) is a function

\[
m : M_k \rightarrow \mathbb{Z}
\]

satisfying

1. \( m(\nu) \geq 0 \) for all \( \nu \), \( m(\nu) = 0 \) a.e. \( \nu \);
2. if \( \nu \) is a real place, then \( m(\nu) = 0 \) or \( 1 \);
3. if \( \nu \) is complex, \( m(\nu) = 0 \).

It is convention to write a modulus as

\[
m = \prod_{\nu} P_{\nu}^{m(\nu)}
\]

We say \( M_1 \) divides \( M_2 \) if \( m_1(\nu) \leq m_2(\nu) \) for all \( \nu \in M_k \).

We often split our modulus up into real and finite parts and write

\[
M = M_{\infty} M_f.
\]

Note that \( M_f \) can be identified with an ideal in \( \mathcal{O}_k \).

Let \( M \) be a modulus. To formulate GCFT in terms of ideals, we need a notion of elements being close to 0 with respect to the modulus. We define \( K_{m_1} \) to be the set of elements \( x \in K^X \) s.t.
More concretely, we see that
\[
\text{ord}_v(x^{-1}) \geq m(v) \iff \frac{\text{ord}_v(m(v))}{x} \leftarrow \frac{m(v)}{x}.
\]
\[
\iff x \mapsto \left\lfloor \frac{m(v)}{\text{ord}_v(m(v))} \right\rfloor.
\]

Write \( S(m) = \sum v : m(v) > 0 \). Observe that for \( x \in \mathcal{K}_{m,1} \),
we have \( \text{ord}_v(x^{-1}) \geq 0 = \text{ord}_v(1) \) and so
\[
\text{ord}_v(x) = \text{ord}_v((x^{-1})^{-1}) = 0,
\]
i.e., \( (x) \in I_{S(m)}^{\mathcal{K}} \). Thus, identifying \( \mathcal{K}_{m,1} \) with its image
in \( I_{S(m)}^{\mathcal{K}} \), we can make the following definition.

**Def.** Let \( m \) be a modulus. The quotient
\[
\mathcal{C}_{\mathcal{K}}^m = \frac{I_{S(m)}^{\mathcal{K}}}{\mathcal{K}_{m,1}}
\]
is called the **ray class group modulo** \( m \).

The following theorem helps to compute ray class groups.

**Theorem 2.12:** Let \( m \) be a modulus of \( \mathcal{K} \). There is an
exact sequence
\[
0 \to \mathcal{C}_{\mathcal{K}}^m / \mathcal{K}_{m,1} \to \mathcal{K}_{S(m)} / \mathcal{K}_{m,1} \to \mathcal{C}_{\mathcal{K}}^m \to \mathcal{C}_{\mathcal{K}} \to 0
\]
along with canonical isomorphisms

\[
\frac{K^*_{S(m)}}{K_{m+1}} \cong \prod_{v \text{ real}} \left( \frac{\mathbb{Q}^*}{\mathbb{Z}^*} \right) \times \prod_{v \text{ non-real}} \left( \frac{\mathbb{Q}^*}{\mathbb{Z}^*} \right)
\]

\[
\cong \left( \frac{\mathbb{Q}^*}{\mathbb{Z}^*} \right)^{\sum_{v \text{ real}} 1} \times \prod_{v \text{ non-real}} \left( \frac{\mathbb{Q}^*}{\mathbb{Z}^*} \right)^{\sum_{v \text{ non-real}} 1}.
\]

Before we can prove this theorem, we need the following lemma.

**Lemma 2.13:** Let \( S \subseteq \mathbb{Q} \) be a finite set. Any \( x \in K^* \) can be written \( x = \frac{a}{b} \) with \( a, b \in \mathbb{Q} \cap K^* \).

**Proof:** Exercise.

**Proof (Thm 2.12):** We obtain a homomorphism \( \delta^m : K_m \to C_m \).

Now the natural inclusion \( I_{K_m} \to I_{K_l} \). We have natural maps

\[
K_m \to K_{m+1} \cong K^{S(m)} \to K^{S(m)}.
\]

Lemma 2.11 gives that the kernel of \( g \) is \( C_k^{\times} \)

and the cokernel is \( C_k \). The map \( f \) is an injection.

Therefore, the kernel of \( g \circ f \) is \( K_{m+1} \cap C_k^{\times} \)

and the cokernel is \( C_m \). Thus, we can apply the kernel-cokernel sequence to \( f \) and \( g \) to obtain

\[
0 \to C_k^{\times} \cap K_{m+1} \to C_k^{\times} \to K_{m+1}^{S(m)} / K_{m+1} \to C_m \to C_k \to 0
\]

It only remains to prove the isomorphisms.
this case we map $x \in K_{n}^{(m)}$ to $\text{sgn}(x)$ where we understand the sign to be the sign of $x$ after it is embedded into $\mathbb{R}$ via the embedding associated to $v$. Let $v$ be a finite place so that $v(\mu) > 0$. Let $x \in K_{n}^{(m)}$ and use Lemma 2.13 to write $x = ab^{-1}$ for $a, b \in \mathcal{O}_{K} \cap K_{n}^{(m)}$. We map $x$ to $ab^{-1}$ in $(\mathcal{O}_{K}/\mathcal{m}(v))^\times$. Note that this makes sense because $a, b$ are both relatively prime to $\mathfrak{p}_v$. The strong approximation theorem gives that the map

$$K_{n}^{(m)} \rightarrow \prod_{\text{red } \mathfrak{m}(v) > 0} \left( \mathcal{O}_{K}/\mathcal{m}(v) \right)^\times$$

is surjective and the kernel is easily seen to be $K_{n,1}$.

The Chinese Remainder Theorem then gives:

$$\prod_{\text{red } \mathfrak{m}(v) > 0} \left( \mathcal{O}_{K}/\mathcal{m}(v) \right)^\times \cong \left( \mathcal{O}_{K}/\mathfrak{m}(v) \right)^\times,$$

which completes the proof.

**Exercise:** Use the previous theorem to show $C_{m}^{(n)}$ is a finite group of order

$$h_{K} \left[ \mathcal{O}_{K} : \mathcal{O}_{K}\cap K_{n,1} \right] \mathcal{O}_{K}^{*} \text{Nm}_{v} \left( \mathfrak{m}_{v} \right) \prod_{\text{red } \mathfrak{m}(v) > 0} \left( 1 - \text{Nm}(\mathfrak{m}_{v}) \right).$$
where \( r = \# \text{ root, } m(v) > 0 \) and \( \text{ord } k \) is of course the order of \( C_k \).

**Example:** 0 Let \( M = \mathbb{Q} \), then \( C_k^M = C_k \). Annihilate \( \text{Im } k \) and \( K_{n1} = K^x \) in this case.

2. Let \( M \) be the product of the real places in \( \text{M} k \). Then the situation \( C_k^M \) is referred to as the maximal class group. In this situation we have \( K_{n1}^M = K^x \) (recall this is defined with \( v \Rightarrow v^M \)), and \( K_{n1} \) is the set of totally positive elements, i.e., the elements that are positive under all real embeddings. This shows that \( K_{n1}^M / K_{n1} = K^x / K_{n1} = \prod_{v \in \text{red}} \mathbb{Q^+} / 2 \). Thus, the kernel of the map \( C_k^M \to C_k \) is the set of possible signs modulo three signs arising from the units.

**Exercise:** 0 Compute the order of the maximal class group of the fields \( \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{15}) \), and \( \mathbb{Q}(\sqrt{15}) \).

2. Use the above exact sequence to give a very simple computation on \( C_k^M \) for \( m = (m) \) and \( m = \infty (m) \).

(Recall you did this computation by hand at the start of last term!)

We are now in a position to define the global Euler map in terms of ideals and gain the main theorems.
At this point it is easy to give the connection between idee theory and idee theory. We do this before delving any further into the theory. This way the reader may easily phrase the results in whatever language is more comfortable. We have already seen that $C_k = \mathbb{R}_+^k / \mathbb{R}_+^k$ is a quotient of $\mathbb{A}_k^\times$. We now show the same thing for $C_k^n$ given a modulus $m$.

Let $m$ be a modulus and $p_v$ a prime set $m \equiv 0$.

Define

$$W^m_k(p_v) = \sum_{v \in \mathbb{R}_+^k} [1 + p_v^{m(v)}]^{-1}.$$  

We have that $W^m_k(p_v)$ is a neighborhood of $1$ in $K_v^\times$ and

$$K_{m,1} = K_v^\times \cap \prod_{m(v) \equiv 0} W^m_k(p_v).$$

Let

$$M^x_{k,m} = \prod_{m(v) \equiv 0} W^m_k(p_v) - \prod_{m(v) \equiv 0} K_v^\times$$

where $\prod'$ denotes the restricted product with $x_v \in K_v^\times$ for a.e. $v \in m(v) \equiv 0$.
\[ W_{K}^{m} = \prod_{v \mid \infty} W_{K}^{m}(v) \times \prod_{v \mid \text{finite}} K_{v}^{\times} \times \prod_{v \mid \text{finite}} \mathbb{C}_{v}^{\times}. \]

Exercise: Show that \( A_{k, m}^{x} \cap K^{x} = K_{m, 1} \), where we consider \( K^{x} \) and \( K_{m, 1} \) as acting in \( A_{k}^{x} \) via the usual diagonal embedding.

**Prop. 8.14:** Let \( m \) be any modulus of \( K \).

1. The map

\[ A_{k, m}^{x} \rightarrow \mathbb{I}_{K}^{x(m)} \]

\[ (x_{v}) \rightarrow \prod_{v \mid \text{finite}} \beta_{v}(x_{v}) \]

defines an isomorphism

\[ A_{k, m}^{x} \rightarrow \mathbb{I}_{K}^{x(m)} \]

\[ A_{k, m}^{x} / K_{m, 1} \cdot W_{K} \rightarrow \mathbb{I}_{K}^{x(m)} / K_{m, 1} = C_{K}^{m}. \]

2. The inclusion

\[ A_{k, m}^{x} \rightarrow A_{k}^{x} \]

defines an isomorphism

\[ A_{k, m}^{x} / K_{m, 1} \rightarrow A_{k}^{x} / K_{x}. \]

Before proving this proposition, note that combining the two parts gives the desired realization of \( C_{K}^{m} \) as a group of
Proof (Prop 2.14): 1. We have a pair of maps

\[ K_{m,1} \xrightarrow{f} \mathbb{A}^{x}_{K,m} \xrightarrow{g} \mathbb{A}^{x}_{K} \]

so that \( f \) is injective, \( g \) is surjective, and the kernel of \( g \) is \( W_{K}^{m} \). Thus, we use the kernel-extension sequence to obtain the exact sequence

\[ W_{K}^{m} \to \mathbb{A}^{x}_{K,m}/K_{m,1} \to \mathbb{A}^{x}_{K} \to 1. \]

This proves 1. 

3. It is clear that the kernel of the map

\[ \mathbb{A}^{x}_{K,m} \to \mathbb{A}^{x}_{K}/K \]

is \( K^{r} \cap \mathbb{A}^{x}_{K,m} = K_{m,1} \). Thus, we obtain an injection

\[ \mathbb{A}^{x}_{K,m}/K_{m,1} \to \mathbb{A}^{x}_{K}/K. \]

To obtain surjectivity, we apply a weak approximation (in a very weak form). Let \((x) \in \mathbb{A}^{x}_{K}\). Choose \(y \in K\) so that \(xy \in W_{K}(x)\) for all \(y \in S\). This is possible since \(K^{r}\) is dense in \(\mathbb{A}^{x}_{K,T}\) as long as \(T\) contains at least one place, or just pick some place outside \(S\). Then \( (xy) \in \mathbb{A}^{x}_{K,m} \) and maps to \((x) \in \mathbb{A}^{x}_{K}/K\). 

\( \square \)
This once again shows how the more natural statement of global class field theory are in comparison to the usual theory. The set $W^m_K(p_v)$ arises naturally in the idele theory as the image of the local norm maps. Thus, when we phrase the idele formulation of GCF as

$$\hat{\mathbb{A}}^\times / K^\times \text{Nm}(\mathbb{A}_L^\times) \xrightarrow{\sim} \text{Gal}(L/K),$$
we are led to considering set of the form $W^m_K(p_v)$. More specifically, we will see that $W^m_K$ arises in $\text{Nm}(\mathbb{A}_L^\times)$ when the $S$ is chosen appropriately. This also gives $K_0$ more naturally as $K^\times \cap \bigcap_{v \notin S} W^m_K(p_v)$.

We also see that we could have defined the Ray class group $C^m_K$ as $\overline{W^m_K} \cdot \mathbb{A}_L^\times \subseteq \mathbb{C}_K^{(K)}$, and we insisted to avoid the idele theory all together. We now continue with Ray class fields in the idele context.

Prop 3.85: The open subgroups of finite index of $C^m_K$ are precisely those subgroups containing $K^\times \mathbb{A}_L^\times / K$ for some $m$. 
Proof: First we note that \( K^r W_k^{m^r} / k^r \) is open in \( G_k \) because \( W_k^{m^r} \) is open in \( M_k \). Observe that \( W_k^{m^r} \) is contained in

\[
W_k^{m^r} = \bigcap_{V \in \mathcal{F}} K^r V \times \bigcap_{V \in \mathcal{F}} U^r V.
\]

(\( = U_{v_{m^r}} \) from Chapter 1). We know that \( M_k / k^r W_k^{m^r} = G_k \) and so \( \left[ G_k : k^r W_k^{m^r} \right] = h_k < \infty \). This gives:

\[
\left[ G_k : k^r W_k^{m^r} \right] = h_k \cdot \left[ K^r W_k^{m^r} : k^r W_k^{m^r} \right]
\]

\[
\leq h_k \cdot \left[ W_k^{m^r} : W_k^{m^r} \right]
\]

\[
= h_k \cdot \prod_{V \in \mathcal{F}} \left[ \mathbb{V}_k : W_k^{m^r} (M) \right] \prod_{V \in \mathcal{F}} \left[ K^r : \mathbb{R}_k \right],
\]

which is finite. Since \( K^r W_k^{m^r} \) is open in the completion of \( G_k \) by finitely many, non-trivial, open cosets, we get that \( K^r W_k^{m^r} \) is dense in \( G_k \). Thus, any group containing \( K^r W_k^{m^r} \) is also open of finite index, being the union of finitely many cosets of \( K^r W_k^{m^r} \). Now just recall closed and open coincide in the topological group setting.

Conversely, let \( N \) be an arbitrary open subgroup of finite index in \( G_k \). We have that the preimage of \( N \) in \( M_k \) is also open and so contains a

subset of the form

\[
\bigcap_{V \in \mathcal{F}} V_k \times \bigcap_{V \in \mathcal{F}} U^r V \times \bigcap_{V \in \mathcal{F}} K^r.
\]
where \( V \) here denotes an open neighborhood of \( \frac{1}{m} \) in \( K^x \), and \( \mathbb{G} \)

is a compact group which we sometimes denote by \( \mathbb{G} \).

If \( V \) is open, we may choose \( N = 1 + \pi m \) for some integer \( m \) since these forms a fundamental system of nbds around \( 1 \).

If \( V \) is real, we may choose \( N = \mathbb{R}_{>0} \). The open set \( V \) thus generates \( \mathbb{R}_{>0} \) (connected component of \( 1 \)). Similarly, for \( V \) complex we get that \( V \) generates \( K^x \). Thus, the

subgroup generated by \( V \) is of the form \( W_k^m \) for a

module \( m \), and so \( N \) contains \( K^x W_k^m / k^x \), as desired.

**Def:** Given a module \( m \), we call \( K^x W_k^m / k^x \) a congruence subgroup modulo \( m \).

The previous proposition combined with the Inertial Theorem (Thm 2.7) gives that associated to a congruence subgroup modulo \( m \) is

a finite abelian extension \( K^m \) of \( K \) so that \( N = K^x W_k^m / k^x \).

Then

\[
\mathbb{G}_K / N \cong \text{Gal}(K^m/K).
\]

This field \( K^m \) is the Ray class field modulo \( m \).
Let $m, m_1$ be modules so that $m | m_1$. Then it is not hard to see that if we write $N_k^{m_1} = K^x W_k^m K^x$, then $N_k^{m_1} \subseteq N_k^m$. This shows, via Helou theorem, that $K^m \subseteq K^{m_1}$. As a corollary of GCFT and the discussion we obtain:

**Kronecker-Weber Theorem:** Let $K/Q$ be a finite abelian extension. Then there exists $m \in \mathbb{Z}_{>0}$ so that $K \subseteq \mathbb{Q}(\sqrt[2^m]{\mathbb{Z}_{>0}})$.

**Proof:** Recall that for the module $m = m \in \mathbb{Z}_{>0}$, we have $\mathbb{C}_m = (\mathbb{Q}/\mathbb{Z})^2 \cong \text{Gal}(\mathbb{Q}(\sqrt[2^m]{\mathbb{Z}_{>0}})$. This shows that $\mathbb{Q}^{m} = \mathbb{Q}(\sqrt[2^m]{\mathbb{Z}_{>0}})$. Moreover, any module $m' \in \mathbb{Z}_{>0}$ that divides $m$ for some integer $m \in \mathbb{Z}_{>0}$, and so we have $\mathbb{Q}^{m'} \subseteq \mathbb{Q}^m = \mathbb{Q}(\sqrt[2^m]{\mathbb{Z}_{>0}})$, as claimed. \[\square\]

More generally, we have:

**End 2016:** Every finite abelian extension $L/K$ is contained in $K^m$ for some module $m$. 
Though at this point we have a good understanding of the correspondence between open subgroups of infinite index in \( \Gamma \) and finite abelian extensions \( K \), we still have no really decisive information on the ramification of the field \( L \). We do know that if \( N^m \triangleleft N_m(\mathbb{Q}_p) \), then we must have \( L/K \) is unramified for all places \( v \) with \( m(v) > 0 \). This follows from our definition of the global Artin map and the relations with the local Artin maps. Namely, we see that \( \mathbb{C} \) is kept for \( v \) with \( m(v) > 0 \), but this is equivalent to \( L/K \) being unramified at \( v \). (Recall \( \psi_v(\mathbb{Q}_p) \to \text{Im} \gamma_v \) and the criteria defined here.)

However, we do not yet know about the places \( v \) with \( m(v) \leq 0 \).

The study of these places leads to the notion of the conductor, which we first define for local fields. One should note that it can't simply be that \( L \) is ramified at the places that divide \( m \) for \( L \subset K^m \). This is because we saw before that if \( m \mid m' \), then \( K^m \subset K^{m'} \) and so we would have to decide which \( m \). This is essentially what turns out to be the conductor.
Def: Let $L/K_v$ be a finite abelian extension of number fields. Let $n \in \mathbb{Z}_{\geq 0}$ be the smallest integer s.t.

$$1 + \mathfrak{f}^n \subseteq \text{Nm}_{L/K_v} L^\times.$$

The ideal $\mathfrak{f} = \mathfrak{f}^n_v$ is called the conductor of $L/K_v$.

Prop 2.17: A finite abelian extension of number fields $L/K_v$ is unramified iff $\mathfrak{f} = 1$.

Proof: First suppose that $L/K_v$ is unramified. We can use the fact that this implies the norm map $\text{Nm}_{L/K_v}: \mathbb{Q}_\tau^\times \to \mathbb{Q}_\tau^\times$ is onto.

(See Prop 5.11 in [CFR notes].) Thus, $\mathbb{Q}_\tau^\times \subseteq \text{Nm}_{L/K_v} \mathbb{Q}_\tau^\times$ and so $\mathfrak{f} = 1$.

Assume now that $\mathfrak{f} = 1$. Let $n = [L : K_v]$. Then $\mathfrak{f}^n_v \subseteq \text{Nm}_{L/K_v} L^\times$

because $K_v^{\text{ab}}/\text{Nm}_{L/K_v} L^\times \cong \text{Gal}(K_v^{\text{ab}}/K_v)$, and since $\# \text{Gal}(K_v^{\text{ab}}/K_v) = n$

we must have $\mathfrak{f}^n_v$ map to $1$ in $\text{Gal}(L/K_v)$, i.e. it is in $\text{Nm}_{L/K_v} L^\times$.

Let $K_n$ be the unramified extension of $K_v$ of degree $n$. Then we have

$$\text{Nm}_{K_n/K_v} K_n^\times = \left( \mathbb{Q}_\tau^\times \right)^n \times \mathbb{Q}_\tau^\times$$

And since $\left( \mathbb{Q}_\tau^\times \right)^n$ and $\mathbb{Q}_\tau^\times$ are both contained in $\text{Nm}_{L/K_v} L^\times$, we have

$$\text{Nm}_{K_n/K_v} K_n^\times \subseteq \text{Nm}_{L/K_v} L^\times.$$
Def: Let \( L/K \) be a finite abelian extension of number fields.

The conductor \( f \) of \( L/K \) (or of \( Nm_{L/K} \)) is the gcd of all the moduli \( m \) s.t. \( L \subset K^m \) (i.e. \( \frac{K^m}{L} \) is unramified).

This definition of the global conductor shows that \( L^f \) is the smallest Ray class field of \( K \) that contains \( L \). This gives that it makes sense to talk about the ramification of primes in \( L \) in terms of the places dividing \( f \).

Warning: In general the conductor of \( K^m \) is not necessarily \( m \! m \)!

The local conductors defined above are usually easier to calculate than the global conductor. For \( v \) an infinite place of \( K \), or an infinite place of \( L \) dividing \( v \), recall that we call \( v \) unramified if \( v \) is an real and unramified otherwise. This corresponds to the notation in the local theory.
that \( E/\mathbb{Q} \) is ramified where \( \mathbb{Q}/\mathbb{Q} \) is clearly not ramified.

As such, we define the local conductor \( f_v \) of \( L/\mathbb{Q}_p \) at an infinite place \( v \) to be \( v \) if \( L/\mathbb{Q}_p \) is ramified and \( 1 \) otherwise. With this convention we see Prop 3.17 holds for each, prime as well.

**Prop 3.18:** Let \( L/\mathbb{Q} \) be a finite abelian extension with conductor \( f \).

For each place \( v \in M_{\mathbb{Q}} \), let \( f_v \) be the local conductor, i.e., the conductor of \( L_v/\mathbb{Q}_v \). We have

\[
f = \prod_v f_v.
\]

**Proof:** We begin by reducing this to a local calculation. Let

\[ N = N_{L_{\mathbb{Q}}/\mathbb{Q}} \] and let \( m = \prod p^{n(v)} \) be a modulus.

Recall from above that

\[ K^* L_{\mathbb{Q}}^m \leq N \iff f \mid m. \]

It is clear that

\[
\prod_v f_v \mid m \iff f_v \mid p^{n(v)} \quad \forall v.
\]

Thus, in order to show \( f = \prod f_v \) it is enough to show that

\[ K^* L_{\mathbb{Q}}^m \leq N \iff f_v \mid p^{n(v)} \quad \forall v. \]
**Claim:** \( N \cap K^* = N \cap_{\phi_0} L^* \)

**Proof:** Recall that we have the commutativity of the local and global action maps, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
K^* & \xrightarrow{\psi} & \text{Gal}(L/K) \\
\downarrow & & \downarrow \\
M^*_K & \xrightarrow{\phi_0} & \text{Gal}(L/K)
\end{array}
\]

where \( K^* \rightarrow M^*_K \) and \( \min \chi \rightarrow \langle x_\nu \rangle = (1, \ldots, 1, x_\nu, 1) \).

Suppose \( x_\nu \in N \cap_{\phi_0} L^* \). Then we have \( \phi_0(x_\nu) = 1 \) and so \( \phi_{1/k}(x_\nu) = 1 \). Thus, \( x_\nu \in N \). Clearly \( x_\nu \in K^* \) and so we obtain \( N \cap_{\phi_0} L^* \subseteq N \cap K^* \).

Now let \( x \in N \cap K^* \). The fact that \( x \in N \) gives that there exists \( y \in M^*_K \) s.t. \( N \cap y = x \) (mod \( K^* \)).

On the other hand, \( x \in K^* \) means \( x = \langle x_\nu \rangle \). Since we are establishing an equality in \( M^*_K \), we have this is all up to \( K^* \) equivalence of \( x \), i.e., \( x \in K^* \) so that \( N \cap y = a \langle x_\nu \rangle \) with equality in \( M^*_K \).

Thus, we have \( N \cap y = (a, a, \ldots, a, ax_\nu, a) \) and so \( a \) is in \( N \cap_{\psi_0} L^* \) for all \( \psi \neq \psi_0 \). However, we then saw that

\[
1 = \phi_{1/k}(a) = \prod_{\psi \neq \phi_0} \phi_\psi(a) = \phi_0(a)
\]

and so \( a \in N \cap_{\phi_0} L^* \). Thus, \( x_\nu \in N \cap_{\phi_0} L^* \) as well which finishes the proof. \( \square \)
We may apply this claim to see:

\[ \mathcal{K}^x \mathcal{W}_K^m / \mathcal{K}^x \cong \mathbb{N} \iff (x \in \mathcal{W}_K^m \Rightarrow x \in \mathbb{N}) \quad \text{for } x \in \mathcal{W}_K^x \]

when we write \( x \) to denote \( x \pmod{K^x} \). This in turn

is equivalent to

\[ (x_v = 1 \pmod{p_v^{m(v)}} \forall v \text{ s.t. } x_v > 0 \quad \forall v \in \text{rad} / \text{min} \Rightarrow (x_v) \in \mathbb{N} \cap K_v^x = \mathbb{N} \cap W_{K_v^x}^x) \]

\( \forall v \).

\( \Leftarrow \)

\( (x_v \in \mathcal{W}_K^m) \Rightarrow x_v \in \mathbb{N} \cap W_{K_v^x}^x \quad \forall v \)

\( \Rightarrow \mathcal{W}_K^m(v) \subseteq \mathbb{N} \cap W_{K_v^x}^x \quad \forall v \)

\( \Rightarrow \mathcal{W}_K^m(v) \cap p_v^{m(v)} = \emptyset , \quad \forall v \)

where we have set \( \mathcal{W}_K^m(v) = \emptyset \) if \( v \nmid \infty \), \( m(v) = 0 \) and

\( \mathcal{W}_K^m(v) = K_v^x \) if \( v \mid \infty \), \( m(v) > 0 \).

Immediately we can combine this Proposition along with

Proposition 2.17 to deduce the following corollary.

**Corollary 3.19:** Let \( L/K \) be a finite cyclic extension and

\( \mathfrak{f} \) the conductor. Then we have \( \mathcal{W}_E \mathcal{M}_x \) is ramified

in \( L \) iff \( \mathfrak{f} \mid \mathfrak{f} \).

\[ \begin{array}{c}
\text{GCFR} \\
\text{p. 66}
\end{array} \]
Let $K$ be a number field and consider the modules $m$-1. Clearly, we have that the conductor of $K^m/K$ is 1 in this case. Thus, the field $K^m$ is unramified at every place. We call this field the \textit{Hilbert class field} of $K$.

It is the maximal unramified abelian extension of $K$.

The fact that all places are unramified shows that

$$\prod_{v | \infty} K_v^* \times \prod_{v | \infty} \mathbb{Q}_v^* \subseteq Nm_{K/K^m}(K^m)^*$$

However, we always have $Nm_{K/K^m}(K^m) \subseteq \prod_{v | \infty} K_v^* \times \prod_{v | \infty} \mathbb{Q}_v^*$ and so we must have equality. Thus, for $K^m$ a Hilbert class field, we have

$$\frac{\mathbb{A}_K^*}{K^*(\prod_{v | \infty} K_v^* \times \prod_{v | \infty} \mathbb{Q}_v^*)} \cong \text{Gal}(K^m/K).$$

However, we have seen before that

$$\frac{\mathbb{A}_K^*}{K^*(\prod_{v | \infty} K_v^* \times \prod_{v | \infty} \mathbb{Q}_v^*)} \cong \mathcal{C}_K$$

and so we obtain $\text{Gal}(K^m/K) \cong \mathcal{C}_K$.

It is customary to write $H_K$ for the Hilbert class field of $K$. 


Exercise: Prove the $G_{L_1}$ case of strong approximation

for a number field $K$, i.e., show that if $h_K = 2$

then

$$\mathcal{A}_K^* = K^*(\prod_{v \mid 150} K_v^* \times \prod_{v \mid 1500} L_v^*).$$

Consider again the modulus $m = \prod_{v \mid n} v$. Recall we defined the associated Ray class group $C_m^K$ as the narrow class

group of $K$. Let $H_K^+$ be the associated Ray class field. It is not difficult to see that in this particular case we have $S = m$. Write $H = H_K^+$ to ease notation for the rest of the argument. We claim that

$$\frac{1}{\mathcal{A}_K^*} K^*(\prod_{v \mid \text{real, } K_v^*} \prod_{v \mid \text{complex, } K_v^*} \prod_{v \mid 1500} L_v^*) \cong \text{Gal}(H_K^+/K).$$

We know that

$$\frac{1}{\mathcal{A}_K^*} K^* \text{Nm } H^* \cong \text{Gal}(H_K^+/K)$$

and that

$$\left(\prod_{v \mid \text{real, } K_v^*} \prod_{v \mid \text{complex, } K_v^*} \prod_{v \mid 1500} L_v^*\right) \leq \text{Nm } H^*.$$

The only possible obstruction to equality here would be if

for real $v$ s.t. $\text{Nm } (H_v^* = K_v^*)$. Suppose this happen.
Then we have that \( \phi_{H/K}(K^x) = 1 \), which gives \( \phi_{v}(K^x) = 1 \) when we raise \( K^x \) to \( \mathbb{A}_k^x/K^x \) for the global action map.

But this gives \( \phi_{v} = 1 \Rightarrow \text{Gal}(H/K) = 1 \). However, this implies \( H = K \), which contradicts \( H/K \) being ramified at all real places. Thus, we have the result that

\[
\mathbb{A}_k^x / \mathbb{K}^x \left( \prod_{\nu \in \mathbb{K}^x \setminus \mathbb{K}_0} K^x_{v} \right) \cong \text{Gal}(H^x/K).
\]

Section 2.3: Statement in terms of ideals and some computations:

In this section we conclude this chapter by working out what the main theorems of GCFT say in terms of ideals. We then give a couple of numerical calculations applying what we have done in this chapter.

Let \( L/K \) be a finite abelian extension of number fields. Recall that for \( \nu \) a prime of \( K \) that is unramified in \( L \), there is a well-defined Frobenius automorphism, denoted \( (\nu, L/K) \), that generates the decomposition group \( D(\nu/K) \) when we work with \( \nu \). Let \( M \) be a modulus on \( L \subset K^m \). We know such a modulus always exists. Let \( f \) be the conductor.
of \( L/K \), we have that if \( m \) and \( n \) any place \( v \in \mathcal{M}_L \) s.t. \( v \mid m \) must be an unramified prime. This allows us to define a map

\[
\psi_{L/K} : I_K^{S(m)} \longrightarrow \text{Gal}(L/K)
\]

\[
\sigma_L \mid \prod_{v \mid m} \mathcal{O}_L(v) \longrightarrow \prod_{v \mid m} (\mathcal{O}_L(v)^*, m). \]

This map is a homomorphism, and it is clear that

\[
\psi_{L/K} (\sigma_L) = \phi_{L/K} (\sigma_L)
\]

where \( \sigma_L = (1, \ldots, 1, x_v, 1, \ldots) \in \mathcal{M}_L \). We call the map \( \psi_{L/K} \) the Artin symbol. The following theorem will allow us to transfer the ideal results to statements in terms of ideals.

**Theorem 2.20:** Let \( L/K \) be a finite abelian extension of number fields, and let \( m \) be a modulus so that \( L \subset \mathcal{M}_m \). The Artin symbol induces a surjective homomorphism

\[
\psi_{L/K} : C_L^m \longrightarrow \text{Gal}(L/K)
\]

with kernel given by \( \text{Ker} \psi_{L/K} = \text{NM}_{L/K} I_L^{S(m)} / K_m \), where we consider \( S(m) \) in \( L \) places in \( M_L \) over the places dividing \( m \) in \( M_L \) when it is used in \( I_L \). Moreover, we have that the following diagram commutes:
\[ 1 \rightarrow \mathbb{N} \rightarrow \mathbb{C}_K \rightarrow \mathbb{C}_K \rightarrow \text{Gal}(L/K) \rightarrow 1 \]
\[ 1 \rightarrow (\mathbb{N} \times \mathbb{C}_K)_{K_m} \rightarrow \mathbb{C}_K^m \rightarrow \text{Gal}(L/K) \rightarrow 1. \]

\textbf{Proof:} We have already shown that given a modulus \( m \), if we set \( N_k^m = K^w W_k^m \) then

\[ \phi: \mathbb{C}_K^m \rightarrow \mathbb{C}_K \]

\[ (x_v) \mapsto \prod_{v \in \mathfrak{p}} \phi_v(x_v). \]

Thus we have a commutative diagram.

\[ \begin{array}{ccc}
\mathbb{C}_K^m \& \phi_{L/K} \\
\downarrow \psi \& \downarrow \text{id} \\
\mathbb{C}_K \& \text{Gal}(L/K) \\
\end{array} \]

We need to show that \( f = \psi_{L/K} \).

Let \( v \in \mathfrak{p}_k \) with \( m(v) = 0 \) while \( \langle \alpha_v \rangle \) for the ideal \( (1, \ldots, 1, \alpha_v, 1, \ldots) \) and \( \overline{\langle \alpha_v \rangle} \) to denote the class of \( \langle \alpha_v \rangle \) in \( \mathbb{C}_K/N_k^m \). We have that

\[ \phi(\langle \alpha_v \rangle) \equiv \phi_v (\text{mod } K_m). \]

We also have

\[ (f \circ \phi)(\overline{\langle \alpha_v \rangle}) = \phi_{L/K} (\overline{\alpha_v}) = (v, \nu_{L/K}) = \psi_{L/K} (\nu_v). \]

Thus, we have that the map \( f \) arises from \( \psi_{L/K} \) and so \( \psi_{L/K} \) is surjective.
We now need to show that the image of \( \text{Nm}_{\mathfrak{m}^k} \mathcal{O}_L \) under the map \( \psi \) is \( (\text{Nm}_{\mathfrak{m}^k} \mathcal{O}_L^{s(m)})_{k^m} / (k_{m_k}^m) \). Write

\[
\mathfrak{n} = \prod_{m_{nk}^o} \mathfrak{w}_{m_{nk}^o} = \prod_{m_{nk}^o} \left( \prod_{w_{m_{nk}^o}} (g_{m_{nk}^o})^{m_{nk}^o} \right)
\]

This allows us to view \( \mathfrak{n} \) as a modulus for \( L \) as well.

**Exercise:** Prove that \( \mathcal{O}_L \cong (\mathbb{A}_{\mathfrak{m}^k})_{L, \mathfrak{m}^k, \psi} \).

This gives that the elements of \( \text{Nm}_{\mathfrak{m}^k} \mathcal{O}_L \cong \text{Nm}_{\mathfrak{m}^k} \mathbb{A}_{\mathfrak{m}^k} (\mathbb{A}_{\mathfrak{m}^k})_{L, \mathfrak{m}^k, \psi} \)

are the classes in \( \text{Nm}_{\mathfrak{m}^k} (x) \) for \( x \in \mathbb{A}_{\mathfrak{m}^k} \). We know that the ideles make use of graded componentwise with the \( v \)-th component being \( \prod \text{Nm}_{\mathfrak{m}^k} (x_{m_{nk}^o}) \). From the theory of local fields we know \( \text{Nm}_{\mathfrak{m}^k} (x_{m_{nk}^o}) = f_{m_{nk}^o}^{(x_{m_{nk}^o})} \) and so we have

\[
\text{ord}_v (\text{Nm}_{\mathfrak{m}^k} (x_{m_{nk}^o})) = f_{m_{nk}^o}^{(x_{m_{nk}^o})} \cdot \text{ord}_v (x_{m_{nk}^o})
\]

This gives that the ideles \( \text{Nm}_{\mathfrak{m}^k} (x) \) is mapped under \( \psi \) to

\[
\psi (\text{Nm}_{\mathfrak{m}^k} (x)) = \prod_{m_{nk}^o} \prod_{w_{m_{nk}^o}} f_{m_{nk}^o}^{(x_{m_{nk}^o})} \cdot \text{ord}_v (x_{m_{nk}^o})
\]

\[
= \text{Nm}_{\mathfrak{m}^k} \left( \prod_{m_{nk}^o} (g_{m_{nk}^o})^{m_{nk}^o} \right).
\]

Thus, the image of \( \text{Nm}_{\mathfrak{m}^k} \mathcal{O}_L \) under \( \psi \) is precisely the group \( (\text{Nm}_{\mathfrak{m}^k} \mathcal{O}_L^{s(m)})_{k^m} / (k_{m_k}^m) \), as claimed.

One can use this theorem to translate the main theorems of GRT as stated above into statements in terms of ideals.
Corollary 2.21: Let \( K \) be a finite abelian extension of a number field. Let \( m \) be a modulus so that \( L \subseteq K \), and \( \sigma(m) > 0 \) (such a modulus always exists).

We have a canonical isomorphism

\[
\mathcal{H}_K : \mathbb{C}_K / N_K \overset{\sim}{\longrightarrow} \mathbb{G}_L(\mathbb{L}_K)
\]

where \( N_K = (N_{m(L)}(L_s))_{K_{m(L)}} / L_{m(L)} \).

Corollary 2.22: Let \( m \) be a modulus of \( K \) s.t. \( 0 < \sigma(m) \).

There is a bijection between subfields of \( K^m \) and subgroups of \( \mathbb{C}^m \) containing \( K_m \) given by

\[
L \longleftrightarrow (N_{m(L)}(L_s))_{K_{m(L)}} / L_{m(L)}.
\]

We are also able to completely determine the decomposition of an unramified prime \( v \) of \( K \) in \( L \) in terms of the group theory.

Corollary 2.23: Let \( K \) be a finite abelian extension of a number field of degree \( n \). Let \( \mathfrak{m} \) be a prime that is unramified in \( L \). Let \( m \) be a modulus s.t. \( L \subseteq K^m \) and \( \sigma(m) = 0 \). Let \( f \) be the order of \( v \) mod \( (N_{m(L)}(L_s))_{K_{m(L)}} \) in the Ray class
$G_K$, i.e. the smallest index set $\mathfrak{p}_v \mathfrak{I}^{\infty} \mathfrak{I}_v^{\infty} K_m, \ldots$

Then $\mathfrak{p}_v$ decomposes in $L$ into a product

$$\mathfrak{p}_v \mathfrak{I}^{\infty} \mathfrak{I}_v^{\infty} = \mathfrak{p}_v \mathfrak{I}^{\infty} \mathfrak{I}_v^{\infty}.$$

So $r = n/4$ distinct primes of degree $f$ over $\mathfrak{p}_v$.

**Proof:** Let $\mathfrak{p}_v = \mathfrak{p}_v \ldots \mathfrak{p}_v$. Be the decomposition of $\mathfrak{p}_v$ in $L$.

The fact that $\mathfrak{p}_v$ is unramified gives that each of the primes $\mathfrak{p}_v$ is distinct and the fact that $4/f_4$ is Abelian gives they each have the same degree $f(\mathfrak{p}_v/\mathfrak{p}_v)$, call this degree $f$.

We know that $f = \# D(w)$, which is also the order of Frobenius. We know from use that $\mathfrak{p}_v \rightarrow Frob_v$ under the isomorphism $\psi^{\infty} \psi^{\infty} \psi^{\infty} \psi^{\infty} : \mathcal{O}_K \rightarrow \mathcal{O}_L$.

We conclude this section with a couple of remarkable examples. It would be beneficial to make up a few such examples on your own and work them out.

**Example:** Show that an $x$ degree 13 Galois extension

$$x \mathfrak{p} \mathfrak{I}^{\infty} \mathfrak{I}_v^{\infty} \mathfrak{I}_v^{\infty} \rightarrow \mathcal{O}_L$$

is $\mathfrak{p}$ ramified only at $\mathfrak{p}$. 

---

The page seems to be discussing the decomposition of primes in a Galois extension, discussing the properties of unramified primes, and providing an example of a specific degree extension.
Suppose $K/Q$ is a degree 13 extension that is Galois and ramified only at $5$. We have $\text{Gal}(K/Q) \cong \mathbb{Z}/13\mathbb{Z}$, so $K/Q$ is necessarily abelian.

The fact that $K$ is ramified only at $5$ implies the conductor must be of form $\mathfrak{f} \cdot 5^n$, with $\mathfrak{f} \neq \mathfrak{a}$. We have that

\[ \text{Gal}(K/Q) \cong \mathbb{Z}/13\mathbb{Z} \times \mathbb{Z}/\mathfrak{f} \mathbb{Z} \times \mathbb{Z}/5^n \mathbb{Z}. \]

The fact that $\mathfrak{f}$ is the conductor of $K/Q$ gives that $W_{\mathfrak{f}} \subseteq N_{\mathfrak{a}} A_k^\times$ and so

\[ \frac{A_k^\times}{\mathbb{Q}^\times W_{\mathfrak{f}}} \longrightarrow \frac{A_k^\times}{\mathbb{Q}^\times N_{\mathfrak{a}} A_k^\times}. \]

This gives

\[ 13 = \# \frac{A_k^\times}{\mathbb{Q}^\times N_{\mathfrak{a}} A_k^\times} = \# \frac{A_k^\times}{\mathbb{Q}^\times W_{\mathfrak{f}}}. \]

It only remains to calculate the order of $\frac{A_k^\times}{\mathbb{Q}^\times W_{\mathfrak{f}}}$. We write $A_k^\times = \mathbb{Q}^\times \left( \bigotimes_{p} \mathbb{Z}_p^\times \right)$ and so we have

\[ \left[ \frac{A_k^\times}{\mathbb{Q}^\times W_{\mathfrak{f}}} \right] = \left[ \mathbb{Z}_5^\times : 1+5^n \mathbb{Z}_5^\times \right] = \#(\mathbb{Z}_5^\times)^{*} \cdot \left[ \mathbb{Z}_5^\times : 1+5^n \mathbb{Z}_5^\times \right] \]

\[ = 4 \cdot 5^{n-1}. \]

However, in the case $13 
\mid 4 \cdot 5^{n-1}$, clearly a contradiction. Thus, no such $K$ can exist.

**Example:** We find a finite abelian Galois extension of $K = \mathbb{Q}(\sqrt[13]{3})$ unramified at all finite places.
Let $\omega_1$ and $\omega_2$ be the two real places of $K$. Set $L = L'$

such an extension. Since $L$ is unramified at all finite places we must have

$$L^* = (K_{\omega_1}^*)_{\omega_1} \times (K_{\omega_2}^*)_{\omega_2} \times \bigcap_{v \not\mid \omega} O_v^*$$

is contained in $Nm(A_K^*)$, i.e., it is in the kernel of the

Askin map.

It is not difficult to calculate, for example use

the Minkowski bound from algebraic number theory, that

$h_K = 2$. This gives

$$L^* = K^* (K_{\omega_1}^* \times K_{\omega_2}^* \times \bigcap_{v \not\mid \omega} O_v^*)$$

If $L^* = K^* U$, then there will be no such extensions.

Exercise: Show that $L^* = K^* U \iff \exists \in O_K^*$ with any
desired sign.

In our case, the fundamental units are

$$\varepsilon_1 = 2 + \sqrt{3} > 0$$
$$\varepsilon_2 = 2 - \sqrt{3} > 0$$

and so $L^* \neq K^* U$ so there will be such extensions.
We must compute $\frac{\Delta_{K^x}}{K^x \mathcal{M}}$. The size of this will tell us the degree of the maximal such extension in. We have

$$K^x \left( K_{\mathcal{M}_1}^x, K_{\mathcal{M}_2}^x, \bigcap_{\mathcal{M}_0} \mathcal{M}_0^x \right) \sim \frac{K_{\mathcal{M}_1}^x \times K_{\mathcal{M}_2}^x}{(K_{\mathcal{M}_1}^x)_{\mathcal{M}_2} \times (K_{\mathcal{M}_2}^x)_{\mathcal{M}_1}, \mathcal{O}_{K^x}^x)}$$

$$= \frac{K_{\mathcal{M}_1}^x \times K_{\mathcal{M}_2}^x}{(K_{\mathcal{M}_1}^x)_{\mathcal{M}_2} \times (K_{\mathcal{M}_2}^x)_{\mathcal{M}_1}, (-1, -1), (e, \overline{e}, 1, 1)}$$

$$\sim \frac{K_{\mathcal{M}_1}^x \times K_{\mathcal{M}_2}^x}{(K_{\mathcal{M}_1}^x)_{\mathcal{M}_2} \times (K_{\mathcal{M}_2}^x)_{\mathcal{M}_1}, (1, 1) \text{ and } (e, \overline{e})}$$

$$\mathbb{Z}/Z$$

(But, you must understand all the above isomorphisms!)

Thus, the maximal such extension is a quadratic extension.

Write $L = K(\sqrt{a})$. The fact that $h_K = 1$ implies all the ideal of $K$ are principal ideals. This allows us to take $\alpha$ as a unit for $\mathbb{Z}$. If there is a prime $\mathcal{P}$ dividing $\alpha$, then $\alpha$ would be ramified at the point $\mathcal{P}$.

The unit group is given by $\mathbb{Z} \times \mathbb{Z}$, so our possibilities
for $a$ are $-1, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. Observe that

$$K(\sqrt{3}) = \mathbb{Q}(\sqrt[3]{3}, \sqrt{3}) = \mathbb{Q}(\sqrt[3]{3}, \sqrt[3]{-3}) = K(\sqrt[3]{3})$$

We have

Now $2$ is unramified in $\mathbb{Q}(\sqrt[3]{3})$, but $3$ is ramified in $\mathbb{Q}(\sqrt[3]{3})$ so it is unramified in $K(\sqrt[3]{3})$. But $2$

is the only possible ramification place, so $K(\sqrt{3})$ is

such a field.

**Exercise:** Show $K(\sqrt[3]{3})$ and $K(\sqrt{5})$ are both ramified at $3$.

Thus, $K(\sqrt{5})$ is the field we were looking for.

**Exercise:** Find the maximal abelian Galois extension of $\mathbb{Q}(\sqrt{3})$

of degree prime to $5$ and unramified away from $5$. 20.
Chapter 3: $\mathbb{Z}_p$-extensions

In this short chapter, we give a brief introduction to $\mathbb{Z}_p$-extensions. There is a vast theory of $\mathbb{Z}_p$-extensions known as class field theory. The interested reader should look into class field theory for further information. There are limited good resources on the subject, but two are Washington's *Introduction to Cyclotomic Fields* (more analytic) and Greenberg's *Cyclotomic Fields*. The latter is more algebraic and gives a better introduction in more modern language.

Def: A \textit{$\mathbb{Z}_p$-extension} of a number field $K$ is an extension $K^{\infty}/K$ such that $Gal(K^{\infty}/K) \cong \mathbb{Z}_p$.

Before we begin developing properties of $\mathbb{Z}_p$-extensions, we show such extensions actually exist so we are not wasting our time. We restrict to odd primes $p$ throughout this section. The case of $p = 2$ is handled with the same methods, however it...
requires a little more care and notation. We begin with the case $K = \mathbb{Q}$. Recall that we have
\[
\text{Gal}(\mathbb{Q}(\sqrt[p^n]{\mathcal{R}})/\mathbb{Q}) = \left(\mathbb{Z}/p^n\mathbb{Z}\right)^* \cong \left(\mathbb{Z}/p\mathbb{Z}\right)^* \times \left(\mathbb{Z}/p^n\mathbb{Z}\right).
\]

where $\sqrt[p^n]{\mathcal{R}}$ is a primitive $p^n$-th root of unity. Let $\mathcal{Q}_n$ be the fixed field of $\left(\mathbb{Z}/p\mathbb{Z}\right)^*$, i.e., $\mathcal{Q}_n = \mathbb{Q}(\sqrt[p^n]{\mathcal{R}})^{\left(\mathbb{Z}/p\mathbb{Z}\right)^*}$. We then have
\[
\text{Gal}(\mathcal{Q}_n/\mathbb{Q}) \cong \mathbb{Z}/p^n\mathbb{Z}.
\]

Let $\mathcal{Q}_\infty = \bigcup_n \mathcal{Q}_n$. We have
\[
\text{Gal}(\mathcal{Q}_\infty/\mathbb{Q}) \cong \text{Gal}(\bigcup_n \mathcal{Q}_n/\mathbb{Q}) \cong \lim_{\leftarrow} \text{Gal}(\mathcal{Q}_n/\mathbb{Q}) \cong \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p.
\]

Thus, $\mathcal{Q}_\infty$ is a $\mathbb{Z}_p$-extension of $\mathbb{Q}$. It is known as the

cyclotomic $\mathbb{Z}_p$-extension, as it arises from cyclotomic fields.
Beklo theory, gain that

\[ \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/n \mathbb{Z} \].

Let \( \mathbb{Q}_n = \bigcup_{n} \mathbb{Q}_n \). We have

\[ \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \text{Gal}(\bigcup_{n} \mathbb{Q}_n/\mathbb{Q}) \]

\[ = \lim_{\leftarrow} \text{Gal}(\mathbb{Q}_n/\mathbb{Q}) \]

\[ = \lim_{\leftarrow} \mathbb{Z}/n \mathbb{Z} \]

\[ = \mathbb{Z}/n \mathbb{Z}. \]

Thus, we see we have a \( \mathbb{Z}/p \)-extension \( \mathbb{Q}_p \). This is called the **cyclotomic** \( \mathbb{Z}/p \)-extension as it comes from the cyclotomic fields. This is actually the only \( \mathbb{Z}/p \)-extension of \( \mathbb{Q} \). We will come back to prove this in a moment.

Let \( K/k \) be a number field. Let \( K_{\infty} = K \mathbb{Q}_n \). We have

\[ \begin{array}{c}
\mathbb{Q} \\
\downarrow \\
K_{\infty} \\
\downarrow \\
K \\
\downarrow \\
\mathbb{Q} \\
\end{array} \]

We know that

\[ \text{Gal}(K_{\infty}/K) \cong \text{Gal}(\mathbb{Q}_n/K \mathbb{Q}_{\infty}). \]
We will show that \( \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q} \) is actually the only \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \).

Let \( K/\mathbb{Q} \) be a number field. We can construct a \( \mathbb{Z}_p \)-extension of \( K \) by using the cyclotomic \( \mathbb{Z}_p \)-extension \( \mathbb{Q}_p \). Let \( K \otimes \mathbb{Q}_p \). (Note people sometimes write \( \mathbb{Q}_p \) or \( K \otimes \mathbb{Q}_p \) instead of \( \mathbb{Q}_p \otimes_{\mathbb{Q}} K \) to make it clear they are referring to cyclotomic \( \mathbb{Z}_p \)-extensions.) We have

\[
\begin{array}{c}
\mathbb{Q}_p \\
\downarrow \quad \downarrow \\
K \quad \mathbb{Q}_p \\
\downarrow \quad \downarrow \\
K \otimes \mathbb{Q}_p \\
\downarrow \\
\mathbb{Q} \\
\end{array}
\]

From Hilbert theory we have

\[
\text{Gal}(K \otimes \mathbb{Q}_p/\mathbb{Q}_p) \cong \text{Gal}(\mathbb{Q}_p/\mathbb{Q}) \otimes \text{Gal}(K/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}_p/\mathbb{Q}).
\]

The fact that \( K \) is a number field gives \( K \otimes \mathbb{Q}_p \) is a finite extension of \( \mathbb{Q}_p \) and so \( \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p) \otimes \text{Gal}(K/\mathbb{Q}) \) is a closed subgroup of \( \text{Gal}(\mathbb{Q}_p/\mathbb{Q}) \cong \mathbb{Z}_p \). Thus, \( \exists \mathbb{Z}_p \)-st
Prop 3.1: Let $K_0/k$ be a $Z_p$-extension. For each $m > 0$, there is a unique field $K_n/k$ s.t. $\text{Gal}(K_n/k) \cong \mathbb{Z}/p^m\mathbb{Z}$ and these fields are the only fields properly between $K$ and $K_\infty$.

Proof: Let $m > 0$. Then $p^mZ_p$ is a closed subgroup of $Z_p$. At $K_n = K_\infty/p^mZ_p$ and observe that

$$\text{Gal}(K_n/k) \cong \text{Gal}(K_\infty/k) / \text{Gal}(K_n/K_\infty) = \mathbb{Z}/p^m\mathbb{Z} \cong Z_p/p^nZ_p \cong Z_p/pZ_p$$

as claimed. To see there are the only proper intermediate fields we just apply Helou's theory which says intermediate fields correspond to closed subgroups and $p^mZ_p$ are the only closed non-trivial proper subgroups of $Z_p$. 

For the cyclotomic $Z_p$-extension $\mathbb{Q}(\zeta_{p^n})$, it is easy to see that the extension $\mathbb{Q}(\zeta_{p^n})$ is unramified away from $p$. This
Prop 3.3: Let $K_{o}/k$ be a $p$-extension and $v \in M_{K_{o}}$ a place at the VFP. The extension $K_{o}/k$ is unramified at $v$.

Proof: Let $I(v) \triangleq \text{Gal}(K_{o}/k) = \mathbb{Z}_{p}$ denote the inertia group of $v$. We wish to show $I(v)$ is trivial. We know that $I(v)$ is a closed subgroup and so $I(v) = p^{n} \mathbb{Z}_{p}$ for some $n$. This immediately shows $I(v) = 1$ for $v_{1}$ as $\#I(v) = 1$ for $v_{1}$. Assume now that $v_{1} \neq v_{2}$. For each $n \geq 0$, we choose $v_{n} \in M_{K_{o}}$ s.t. $v_{n+1} < v_{n}$ and $v_{0} = v$.

Let $K_{v_{0}} = \bigcup_{n} K_{v_{n}}$. We have

$I(v) \triangleq \text{Gal}(K_{v}/K_{v})$.

As usual, write $\mathbb{Z}_{v}$ for the units of $K_{v}$. We know from CRT that

$\mathbb{Z}_{v}^{*} \rightarrow I(v)$.

Assume $I(v) = p^{n} \mathbb{Z}_{p}$ for some $n$. We also have from the unit theorem that

$\mathbb{Z}_{v}^{*} \cong (\text{finite group}) \times \mathbb{Z}_{p}^{*}$.
for some } a \in \mathbb{Z} \text{, where } V(1, \Delta) \in M_k \text{. (Prove this as an exercise. Use } \text{Log}_p \text{.) } \text{Then we have a surjection}

(\text{finite group}) \quad \mathbb{Z}/p^n \mathbb{Z}_p \longrightarrow p^n \mathbb{Z}_p.

However, } p^n \mathbb{Z}_p \text{ is torsion-free and so we must have }

\mathbb{Z}/p^n \mathbb{Z}_p \longrightarrow p^n \mathbb{Z}_p.

But then we have a continuous surjection

\mathbb{Z}/p^n \mathbb{Z}_p \longrightarrow p^n \mathbb{Z}_p \longrightarrow p^n \mathbb{Z}_p/p^n \mathbb{Z}_p.

This implies that } \mathbb{Z}/p^n \mathbb{Z}_p \text{ has a closed subgroup of}

\text{order } p, \text{ which is a contradiction. Then, } \mathbb{Z}/p^n \mathbb{Z}_p = 0 \text{ as claimed.} \quad \blacksquare

We can use this result to easily show that } \mathbb{Q}_p, \text{ the cyclotomic}

\mathbb{Z}/p^n \mathbb{Z}_p \text{-extension of } \mathbb{Q} \text{ is the only } \mathbb{Z}/p^n \mathbb{Z}_p \text{-extension of } \mathbb{Q}.

\text{Cor 3.3: Let } \mathbb{Q}_k/\mathbb{Q} \text{ be any } \mathbb{Z}/p^n \mathbb{Z}_p \text{-extension of } \mathbb{Q}. \text{ Then }

\mathbb{Q}_k = \mathbb{Q} \text{ and the cyclotomic } \mathbb{Z}/p^n \mathbb{Z}_p \text{-extension of } \mathbb{Q}.

\text{Proof: Write } \mathbb{Q}_k \text{ for the intermediate field of } \mathbb{Q}_p \text{ over an extension } \mathbb{Q}_k \mathbb{Q}_p \text{ as constructed in Prop 3.1.}
The Kronecker-Weber theorem says that for each \( n \), \( \mathbb{Q} \)
most \( \mathbb{Q}_n \leq \mathbb{Q}(5^n) \). Write \( m = p_1^{e_1} \cdots p_r^{e_r} \)
with \( e_i \geq 1 \), \( p_i \) distinct primes. We know that
we have

\[
\begin{tikzpicture}
  \node (Q) at (0,0) {$\mathbb{Q}(5^n)$};
  \node (Q1) at (-1,-1) {$\mathbb{Q}(5^{p_1})$};
  \node (Q2) at (1,-1) {$\mathbb{Q}(5^{p_2})$};
  \node (Q3) at (0,-2) {$\mathbb{Q}$};
  \draw (Q) -- (Q1);
  \draw (Q) -- (Q2);
  \draw (Q1) -- (Q3);
  \draw (Q2) -- (Q3);
\end{tikzpicture}
\]

with each \( \mathbb{Q}(5^{p_i}) \) totally ramified at \( p_i \) and
unramified elsewhere. Consider the field \( K = \mathbb{Q}_n \cap \mathbb{Q}(5^{p_i}) \).

Suppose \( p_i = p \). Then we have that \( \mathbb{Q}_n \) is unramified
at \( p \) by Prop 3.2 and \( \mathbb{Q}(5^{p_i}) \) is totally ramified
at \( p \) and so \( K = \mathbb{Q} \). Thus, it must be that
\( p_i = p \) for some \( i \) and \( \mathbb{Q}_n \leq \mathbb{Q}(5^{p_i}) \). Thus,
\( \mathbb{Q}_n \leq \cup \mathbb{Q}(5^{p_i}) \) and so \( \mathbb{Q}_n \leq \mathbb{Q}(5^n) \). Since they
are both \( \mathbb{Q}_p \)-extensions, we obtain equality.

In general more \( \mathbb{Q}_p \)-extensions exist. We are prove
theorem that tells us how many \( \mathbb{Q}_p \)-extensions of a number,
field exist. First, we prove one more easy result on the ramification in \( \mathbb{Z}_p \)-extensions.

**Prop 3.4:** Let \( K \subseteq \mathbb{Q}_p \) be a \( \mathbb{Z}_p \)-extension. There is at least one place that ramifies in this extension. Moreover, there is an \( n \geq 0 \) s.t. every prime that ramifies in \( K \subseteq \mathbb{Q}_p \) is totally ramified.

**Proof:** The fact that the Hilbert class field is the maximal abelian unramified extension of \( K \) and it is a finite extension immediately gives that at least one place must ramify in \( K \subseteq \mathbb{Q}_p \).

The only possible places that can ramify are those over \( p \).

Let \( v_1, \ldots, v_r \) be the places over \( p \) that do ramify. We know that \( I(v_i) \) is a maximal closed subgroup of \( \mathbb{Z}_p \) for \( i = 1, \ldots, r \) and \( n \in \mathbb{N} \) s.t.

\[
I(v_i) = p^n \mathbb{Z}_p.
\]

The fixed field of \( p^n \mathbb{Z}_p \) is \( K_n \); by the definition of \( K_n \) and \( v_i \),

\[
\text{Gal}(K_n/K_{v_i}) = I(v_i).
\]
There, all of the primes of $K_m$ over $\mathfrak{p}$ are totally ramified in $K_{oo}/K_m$. Since the $K_m$'s are ordered by inclusion, if we set $m = \max \{ n_i : i = 1 \ldots d \}$, we have the prime that ramifies in $K_{oo}/K_m$ are totally ramified in $K_{oo}/K_m$.

Write $E_1$ for the set of units of $K$ that are congruent to 1 modulo every prime $\mathfrak{p} \in \mathcal{M}_K$, i.e., the notation $\mathcal{M}_K$. However, we use $E_1$ to ease notation as well as observe standard conventions. Write $U_1$ to denote the local unit congruent to 1 modulo $\mathfrak{p}$ and

$$U_1 = \prod_{\mathfrak{p} \in \mathcal{M}_K} \mathcal{O}_K^{x}. $$

We have a natural embedding

$$\mathcal{O}_K^{x} \hookrightarrow \prod_{\mathfrak{p} \in \mathcal{M}_K} \mathcal{O}_K^{x}$$

$$x \mapsto (x, x, \ldots, x).$$

Note that $E_1$ is the precisely those $x$ with image in $U_1$.

For $x \in \mathcal{O}_K^{x}$, we have that $x^{N_{\mathcal{M}_K}} \in E_1$ and so $E_1$. 

(Continued on the next page...)
is a subgroup of $\mathcal{O}_K^*$ of finite index, i.e., we have that $E_1$ has $\mathbb{Z}_p$-rank $r + r_s - 1$ as well when $r + r_s = [K : \mathbb{Q}^s]$ as usual.

We know that $U_1$ is a $\mathbb{Z}_p$-module under the action $s \cdot x = x^s$ (check this works $U_1$ into a $\mathbb{Z}_p$-module). Let $\bar{E}_1$ be the closure of $E_1$ in the topology of $U_1$. Then $\bar{E}_1$ is a $\mathbb{Z}_p$-module as well. It is natural to ask what the $\mathbb{Z}_p$-rank of $E_1$ is. In general, this is an open problem with the following conjectural answer.

**Leopoldt's Conjecture:** The $\mathbb{Z}_p$-rank of $\bar{E}_1$ is $r + r_s - 1$.

**Theorem:** If $K/q$ is an abelian extension, then Leopoldt’s Conjecture is true.

**Proof:** See §5.6 of Washington.

We move now to the main theorem that we will prove in this chapter. This theorem counts the number of $\mathbb{Z}_p$-extensions of a number field.
Thm 3.5: Let the $\mathbb{Z}_p$-rank of $E$ be $r_1 + r_2 - 1 - s$ with $s \geq 0$. There are $r_1 + 1 + s$ independent $\mathbb{Z}_p$-extensions of $K$, i.e., if we write $\mathbb{Z}_p(K)$ for the compositum of all $\mathbb{Z}_p$-extensions of $K$, then $\text{Gal}(\mathbb{Z}_p(K)/K) \cong \mathbb{Z}_p^{r_1 + 1 + s}$.

Proof: Let $H_K^p$ be the maximal adelic extension of $K$ that is unramified away from $p$. Note that $\mathbb{Z}_p \subset H_K^p$.

Let

$$U_p = \prod_{v \not= p} \mathbb{C}_v^*$$

and

$$U_{p,1} = \prod_{v \not= p} \mathbb{C}_v^* \times \prod_{v = p} K_v^*.$$  

We know from Cor. that $K^* U_{p,1} \subset H_K^p$, $\phi_K \subset K^* N_{K/H_K^p}$.

However, since $H_K^p$ is the maximal adelic extension, we obtain that

$$\text{Ker } \phi_K = \overline{K^* U_{p,1}}$$

when the closure is taken inside the topology of $A_K^\infty$.

Note that $K^* U_{p,1} \subset K^* U_p U_{p,1}$. Thus, we have

$$\text{Gal}(H_K^p/K) \cong \frac{A_K^\infty}{[K^* U_{p,1}]}.$$  

With $N = \overline{K^* U_{p,1}}$ to ease the notation. Observe that we have

$$K^* U_p U_{p,1}/N \cong U_p N/N \cong U_p/U_{p,1} N.$$
Observe that we have

$$U_p = U_1 \times (\text{finite group})$$.

Thus,

$$\left( k^x U_p U_p / N \right) / (\text{finite}) \cong U_1 / U_1 \cap N$$.

We embed $$E_i$$ into $$\mathbb{M}_k^x$$ via the map

$$\tau : E_i \rightarrow U_1 \rightarrow \mathbb{M}_k^x$$

$$x \mapsto (x, 1)$$,

i.e., the element $$x \in E_i$$ maps to the tuple with $$x$$ in all places

where $$\nu | p$$ and 1's elsewhere.

**Claim:** $$U_1 \cap N = \tau(E_i)$$.

**Proof:** Let $$x \in E_i$$. Then we have $$\tau(x) = x \cdot \left( \frac{\mathfrak{p}(x)}{\mathfrak{p}} \right) + k^x U_p$$.

Thus, we obtain $$\tau(E_i) \leq U_1 \cap N$$, and we take closures to obtain $$\overline{\tau(E_i)} \leq U_1 \cap N = U_1 \cap N$$.

The other direction is substantially more difficult.

We may take closures by looking at the intersection over a cofinal system of closed sets containing the set. Write $$U_{\nu_1}$$ for the limit, where the set consists of elements that are congruent to 1 modulo

$$p_1^\infty$$. Let

$$U = \prod_{\nu \mid p} U_{\nu} \times \prod_{\nu \not\mid p} \mathbb{Z}$$.
Thus, we have
\[ K^* U_{x_0} = \cap K^* U_{x_1} U_{x_2} \]
and
\[ T(E) = \cap T(E) U_n. \]

Thus, it is enough to show that
\[ U_n \cap K^* U_{x_1} U_{x_2} \subseteq T(E) U_n, \]
for all \( n \geq 1. \) Let \( x \in x_{x_1} U_{x_2} \subseteq K^* U_{x_1} U_{x_2} \) with \( x \in K^*, u \in U_{x_2}, \)
\( u \in U_n. \) Suppose \( x \in U_1, \) so well. As \( u_x \in U_1 \cap (U_n \cap U_1), \)
we have that \( xu_x \in U_1 \) as well. By definition we know that \( u_x \) has 1 for the component \( \nu \) of \( x, \) and \( x \in U_1 \)
for all \( \nu \) of \( x. \) As \( xu_x \in U_1, \) and \( u_x \) has unit component
for all \( \nu, \) \( xu_x \in U_n \) must be in \( U_{x_1} \) for all \( \nu \) as well. Thus, \( x \in C \) and the fact that it is in \( U_{x_1} \) for all \( \nu \)
implies \( x \in E_1. \) Thus, we have \( \chi u_x \) has component 1 for \( x \in U_{x_1} \) (because that is how \( U_1 \) is defined when we extend \( x \) to \( A_{x_1} \)) and for \( \nu \) of \( x \) we have \( xu_x \in T(E_1). \) Thus,
\[ xu_x \in T(E_1). \]

Thus,
\[ \alpha = xu_x, u_x \in T(E_1) U_n, \]
as we wanted to show. \( \square \)
For large enough \( n \) (namely, \( n > \frac{\epsilon(p)}{p-1} \)), we have that

the \( p \)-adic logarithm gives an isomorphism

\[
U_{1, n} \simeq \mathbb{Z}_p \simeq \mathbb{Q}_p^*.
\]

for \( v \mid p \). However, since a group we have \( \mathbb{Z}_p \simeq U_1 \) and so we

obtain

\[
U_{1, n} \simeq U_1 \quad \text{for large enough } n.
\]

However, we

also have

\[
\mathbb{Q}_p \simeq \mathbb{Z}_p^{\epsilon(p), n(1/p)}.
\]

Thus, \( [k: \mathbb{Q}_p] = \sum \epsilon(p') \delta_{1/p} \), we have

\[
U_1 = (\text{finite}) \times \mathbb{Z}_p^{[k: \mathbb{Q}_1]}
\]

where we have used \( [U_1, U_{1, n}] < \infty \) and

\[
U_1 = \prod_{v \mid p} U_{1, v} = \prod_{v \mid p} U_{1, v}.
\]

Thus, we have

\[
\frac{U_1}{U_{1, n} \cap N} = \frac{U_1}{\tau(E_1)} \simeq (\text{finite}) \times \mathbb{Z}_p^{r_1 + 1 + s}
\]

when we used that \( \tau(E_1) \) has \( \mathbb{Z}_p \)-rank \( r_1 + s - 1 + s \) and

\[
[k: \mathbb{Q}_p] = r_1 + s_1.
\]

Since \( U_p \) differs from \( U_1 \) only by a finite

group, we have

\[
k^* U_p U_{1, n} \simeq U_p U_{1, n} \cap N \simeq \text{(finite)} \times \mathbb{Z}_p^{r_1 + 1 + s}.
\]
We have

$$G_K = \mathbb{A}_K^\times / \mathbb{K}^\times U_p U_{p_1} = (\mathbb{A}_K^\times / N) / (\mathbb{K}^\times U_p U_{p_1} / N)$$

and so $$(\mathbb{A}_K^\times / N) / (\mathbb{Z}_p^{G+1+5}) \approx \text{finite group.} \quad (\text{closed and finite}).$$

This is almost what we want, except we'd like the quotient of $\mathbb{A}_K^\times / N$ by a finite group to be $\mathbb{Z}_p^{G+1+5}$. This would finish the proof. Write $N_1 = N : H$ where $H$ is the finite group.

Then we have

$$\mathbb{A}_K^\times / N_1 \approx \mathbb{Z}_p^{G+1+5}.$$ 

Suppose $F \neq \mathbb{Z}_p(k)$ so $\text{Gal}(F/k) \approx \mathbb{Z}_p^{G+1+5}$. We know that $\mathbb{Z}_p(k) \subset H_k$ and that $\text{Gal}(\mathbb{Z}_p(k)/k) \approx \mathbb{Z}_p^a$ for some integer $a$. However, if $a < r+1+5$, we would obtain a $\mathbb{Z}_p$-ext of $K$ not contained in $\mathbb{Z}_p(k)$, namely, the field coming from the $1 \times 1 \times \ldots \times 1 \times \mathbb{Z}_p$ term, i.e., the fixed field of this.

This is a contradiction, so we would have $F = \mathbb{Z}_p(k)$ and

$$\text{Gal}(\mathbb{Z}_p(k)/k) \approx \mathbb{Z}_p^{r+1+5}. \quad \text{As it only remains to show}$$

$$(\mathbb{A}_K^\times / N) / \text{finite group} \approx \mathbb{Z}_p^{G+1+5}. $$
Let $n$ be the order of the finite group in the equation

$$\left(\frac{\mathbb{A}^n}{\mathbb{N}}\right)^r \cong \mathbb{Z}_p^{r+1+s}.$$ 

Then we have

$$\left(\frac{\mathbb{A}^n}{\mathbb{N}}\right)^r \cong \mathbb{Z}_p^{r+1+s}$$

as $\mathbb{Z}_p$-modules. (Be careful of the switching of exponents here, $\mathbb{A}^n$ is a group under multiplication. As strictly speaking, we should be looking at $\mathbb{A}^n/\mathbb{N}$ modules the homomorphic image of $\mathbb{Z}_p^{r+1+s}$.)

Let $(\mathbb{A}^n/\mathbb{N})_{\text{fin}} = \{ x \in \mathbb{A}^n/\mathbb{N} : x^n = 1 \}$. We claim this is a finite subgroup. Suppose the order is larger than $n$.

This would give two elements of $(\mathbb{A}^n/\mathbb{N})_{\text{fin}}$ the same XOR

in the finite group above, call them $x_1, x_2, \ldots, x_1, x_2 \in (\mathbb{A}^n/\mathbb{N})_{\text{fin}}$

such that $x_1 \oplus x_2 = 0 \in (\mathbb{A}^n/\mathbb{N})_{\text{fin}}$.

Then

$$x_1 - x_2 \in \mathbb{Z}_p^{r+1+s}.$$ 

However, $n(x_1 - x_2) = nx_1 - nx_2 = 0 \mod \mathbb{Z}_p^{r+1+s}$.

$x_1 - x_2 \neq 0$. This is a contradiction as $\mathbb{Z}_p^{r+1+s}$ has no nontrivial torsion, and so $\# (\mathbb{A}^n/\mathbb{N})_{\text{fin}} \leq n$.

Finally, we have $(\mathbb{A}^n/\mathbb{N})_{\text{fin}}$ is clearly closed and

$$\left(\frac{\mathbb{A}^n}{\mathbb{N}}\right)/ (\mathbb{A}^n/\mathbb{N})_{\text{fin}} \cong \left(\frac{\mathbb{A}^n}{\mathbb{N}}\right)^r \cong \mathbb{Z}_p^{r+1+s}.$$
Note that for fields where we know Leopoldt's conjecture, this gives us a precise count on the number of independent $\mathbb{Z}_p$-extensions.

Example: Let $K = \mathbb{Q} (\sqrt{a})$ with $a > 0$. In this case we have $K/\mathbb{Q}$, and Leopoldt's conjecture is known and in $\text{Gal} (\mathbb{Z}_p^{(K)}/\mathbb{Q})$ a $\mathbb{Z}_p$. Thus, at only $\mathbb{Z}_p$-extension in this case is the cyclotomic extension.

(2) Let $K = \mathbb{Q} (\sqrt{a})$ with $a < 0$. The previous theorem gives there are two independent $\mathbb{Z}_p$-extensions in this case.

We know that we always have the cyclotomic $\mathbb{Z}_p$-extension, so there is one of them. We call the other $\mathbb{Z}_p$-extension the anticyclotomic $\mathbb{Z}_p$-extension. In general, when there are $\mathbb{Z}_p$-extensions other than just the cyclotomic $\mathbb{Z}_p$-extension, these other extensions are much harder to study.

We end this chapter with an important theorem on the growth of the $p$-part of class groups in $\mathbb{Z}_p$-extensions. Though the proof is most incredibly advanced, it requires some set-up that we have time...
\textbf{Thm 3.6:} Let \(K/\mathbb{Q}_p \) be a \( \mathbb{Q}_p \)-extension. Let \( p^n \) be the exact power of \( p \) dividing \( \nu_K \). Then there exist integers \( \lambda \geq 0 \), \( \mu \geq 0 \), and \( \nu \) all independent of \( n \), and an integer \( \eta \) such that \( \nu_K = \lambda n + \mu p^n + \nu \).

\textbf{Thm 3.7:} If \( K/\mathbb{Q} \) is abelian, then the \( \mu + \) invariant of \( K^{ab} \) is 0.
§ 4.1 Langlands functoriality (roughly):

In this first section we give a rough idea of Langland's conjecture relating automorphic forms to Galois representations.

We will give (imprecise) definitions of each and brief explanations on why such a result would be an important one. The reason for setting up such a general framework is that we can then see that class field theory is just an instance of this conjecture.

Let \( K \) be a number field and \( L/K \) a finite abelian extension. Let \( \text{GL}_n(L) \) have the usual meaning of \( n \times n \) invertible matrices with entries in \( L \).

**Def:** A (complex) Galois representation is a group homomorphism

\[
p : \text{Gal}(L/K) \to \text{GL}_n(L),
\]

We say it is an \( n \)-dimensional representation. More generally, one can consider Galois representations

\[
p : \text{Gal}(\bar{K}/K) \to \text{GL}_n(L)
\]

or even

\[
p : \text{Gal}(\bar{K}/K) \to \text{GL}_n(E)
\]

for some field (possibly \( p \)-adic) \( E \).
Associated to such a representation is an $L$-function called the Artin $L$-function. It is defined locally at each place of $k$. We will come back to give a more complete description in a future section, so here we just define the local factors for finite unramified places. Let $v : M_k$ be a finite unramified place and $w : M_k$ a place that divides $v$. Let $Fr_v$ be the Frobenius element. Recall that Frobe is well-defined only up to conjugation unless $Gal(M_k)$ is abelian.

Fortunately, this is enough for our purposes. Let

$$L_v(s, \rho) = \det (I - \rho(Fr_v)(Nm v)^{-s})^{-1},$$

Exercise: Check this depends only on the conjugacy class of Frobe.

We then set

$$L(s, \rho) = \prod_v L_v(s, \rho)$$

when we have our really defined $L_v$ for finite unramified $v$. 
We now give a rough idea of the analytic side of things. The automorphism forms. We consider the group

\[ \text{GL}_n(K) \]  

Let \[ Z(\text{GL}_n) \] be the scalar matrices in \[ \text{GL}_n(K) \].

Much as we showed \( H \) (like \( K \)) discretely, one can show

\[ \text{GL}_n(K) \ltimes \text{GL}_n(\mathbb{A}_K) \]  

discretely. Moreover, if we put a Haar measure on \( \text{GL}_n(\mathbb{A}_K) \), the quotient group \[ Z(\mathbb{A}_K) \backslash \text{GL}_n(\mathbb{A}_K) \] has finite volume. Let \( \omega : \mathbb{A}_K^\times \rightarrow \mathbb{C} \) be a unitary character (we will discuss these more later!) Unitary means

\[ |\omega(x)| = 1 \quad \forall x. \]

Let \( L^2(\text{GL}_n(K) \backslash \text{GL}_n(\mathbb{A}_K), \omega) \) be the space of \( \omega \)-measurable functions \( f \) such that

1. \[ f( \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) g) = \omega(c) f(g) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{GL}_n(\mathbb{A}_K) \]

2. \[ \int_{\text{GL}_n(\mathbb{A}_K)} |f(g)|^2 \, dg < \infty \]

We let \( L^2(\text{GL}_n(K) \backslash \text{GL}_n(\mathbb{A}_K), \omega) \) be the subspace of \( L^2(\text{GL}_n(K) \backslash \text{GL}_n(\mathbb{A}_K), \omega) \) satisfying \( \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_n(\mathbb{A}_K) \).
\[ \int f((L^\infty_{\text{rep}})g) \, dx = 0 \quad \forall g \in GL_n(\mathbb{A}_K), \text{is c.c.} \]

This space is referred to as the space of \textit{cusp forms}. There is an action of \( GL(n, \mathbb{A}_K) \) on \( L^2(G(\mathfrak{K})\backslash G(\mathbb{A}_K), \omega) \) given by the right regular representation

\[ g \cdot f(g') = f(g'g). \]

The space of \textit{cusp forms} is \textit{invariant} under this representation and decomposes into an \textit{infinite direct sum} of \textit{invariant} subspaces, \( \phi \rightarrow \pi \) is a \textit{representation} of \( GL_n(\mathbb{A}_K) \) that is \textit{isomorphic} to the representation on one of these \textit{invariant subspaces} (for some \( \omega \)), we call \( \pi \) the \textbf{automorphic cuspidal representation}. The \textit{parameter} \( \omega \) is called the \textbf{central character} of \( \pi \).

If \( \pi \) is a \textit{cuspidal automorphic} rep. on \( GL_n(\mathbb{A}_K) \), then there is an associated \( L \)-function

\[ L(s, \pi) = \prod_{\chi} \left( \frac{(1 - \chi(s)\hbar \omega^{-1})}{(1 - \chi(s)\hbar \omega^{-1})} \right)^{-1} \]

where the \( \chi \)'s are called \textit{Arthur parameters}.
Conjecture (Langlands): Let $p: \text{Gal}(\bar{k}/k) \to \text{GL}_n(k)$ be a Galois representation. Then exists an automorphic representation $\pi$ of $\text{GL}_n(A_k)$ so that

$$L(s, \pi) = L(s, p).$$

Moreover, "operations" on the Galois side should correspond to "operations" on the automorphic side.

One should note that we do not have to restrict to finite extensions $\bar{k}/k$, one would just need to use the Weil group instead of the Galois group in that case.

Not all automorphic representations come from Galois reps, just the nice ones. The power of such a conjecture is when it is known to be true, one can use algebraic techniques on the Galois representation side to obtain information about automorphic forms and one can use analytic information on the automorphic side to obtain information on the Galois side. For example, it is generally easier to study properties such as analytic continuation, functional eqns, etc.
On the other hand, we have natural operations on the Galois side that (tangentially) give operation on the sides that are not so clear. For example, let \( K \leq L \leq E \) be a tower of Galois field extensions. We know that
\[ \text{Gal}(E/K) \text{ is a subgroup of } \text{Gal}(E/L), \] and so given a Galois representation \( \rho : \text{Gal}(E/K) \to \text{GL}_n(E) \), we obtain a Galois representation \( \rho_L = \rho |_{\text{Gal}(E/L)} \). According to the conjecture,
\[ \psi_L \in \text{Rep}_L, \] is the rep. associated to \( \rho_L \), there should be a rep. \( \psi_K \) associated to \( \rho_K \). This rep. is called the Base change of \( \psi_L \) to \( K \) and often denoted \( \text{BC}(\psi) \). This is then a rep. on \( \text{GL}_n(K\mathbb{A}_f) \). The existence of \( \text{BC}(\psi) \) is not known in general even though it is such a simple operation on the Galois side of things!"
In this section we give a more thorough account of the algebraic side of the Langlands conjecture. Let \( \mathbb{L} \) be a finite Galois extension of a number field.

**Def.** A (complex) \( n \)-dimensional representation \( \rho \) of the Galois group \( \text{Gal}(\mathbb{L}/\mathbb{K}) \) is a homomorphism

\[
p : \text{Gal}(\mathbb{L}/\mathbb{K}) \to \text{GL}(V)
\]

for some \( n \)-dimensional complex vector space \( V \).

Note that one can view \( p : \text{Gal}(\mathbb{L}/\mathbb{K}) \to \text{GL}(\mathbb{C}) \) if one chooses a basis of \( V \). If one wishes to consider representations

\[
p : \text{Gal}(\mathbb{L}/\mathbb{K}) \to \text{GL}(V),
\]

one should require that the map be continuous as well.

Let \( \psi : \mathbb{M}_k \to \mathbb{M}_k \), \( k = 0, 1, \ldots, \infty \). Recall the decomposition group is defined by

\[
D(\psi) = \{ \sigma \in \text{Gal}(\mathbb{L}/\mathbb{K}) : \sigma(\psi) = \psi \}.
\]
The inertia group $I(w)$ is defined by the exact sequence

$$1 \to I(w) \to D(w) \to \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_w) \to 1,$$

i.e., it consists of the elements of $D(w)$ that are trivial when restricted to $\mathbb{Q}_p/\mathbb{Q}_w$. You should check that this definition coincides with the definition in terms of the completion $\mathbb{K}_w$ and $K_w$. Recall that $x$ is a canonical generator $\mathcal{X} \to \mathcal{X}_{\mathbb{Q}_p}$ in $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_w)$. Let $F_{\mathbb{Q}_w} \subset D(w)$ be the element mapping to this generator. Note that $F_{\mathbb{Q}_w}$ is well-defined only up to automorphisms modulo $I(w)$. As we should really write $(F_{\mathbb{Q}_w})_{I(w)}$.

Let $V^{I(w)}$ be the maximal quotient of $V$ that is stable under $p(D(w))$ and is fixed by $p(I(w))$, i.e.,

$$p(D(w)) \subseteq \text{GL}(V^{I(w)})$$

and

$$p(I(w)) = \text{id}.$$

Note that since $I(\mathbb{Q}_w) = 1$ if $w/v$ is unramified, we have that $V^{I(w)} = V$ for all but finitely many $w$.

**Def.** We say $p$ is unramified at $w$ if $p(I(w)) = 1.$
\[ L_v(T, p) = \det \left( I_n - Tp(Frob) \right) \]

where \( I_n \) is the \( n \times n \) identity matrix. Note that this is essentially the characteristic polynomial of \( p(Frob) \), and as it is defined by the eigenvalue of \( p(Frob) \). As such, the fact that Frobenius is defined only up to conjugation has no effect here. Also, since \( p(Frob) \) has finite order (since Frobenius), we must have that the eigenvalues are roots of unity, and so \( L_v(T, p) \neq 0 \).

**Def:** The Artin \( L \)-function attached to \( p \) is given by

\[ L(s, p, \chi) = \prod_{\nu \in \mathbb{Q}(p)} L_v(N\nu^{-s}, p)^{-1}. \]

**Lemma 4.1:** \( L(s, p, \chi) \) converges absolutely and uniformly for \( \Re(s) > 1 + \delta \) for every \( \delta > 0 \).

**Proof:** Write

\[ L_v(T, p) = \prod_{i=1}^n (1 - \alpha_i T) \]

with \( \alpha_i \)'s the eigenvalues of \( p(Frob) \) acting on \( V^\mathbb{Q}(p) \). Thus we have
\[ \mathcal{L}_{\nu}(Nm, p)^\mathbb{C} = \prod_{\nu} (1 - \alpha_{\nu} Nmu^{-s})^{-1} = \prod_{\nu} \left( \sum_{k=0}^{\infty} \alpha_{\nu} (Nm)^{-sk} \right) \]

Observe we have

\[ |\mathcal{L}(s, p)| \leq \prod_{\nu} d \prod_{\nu} \sum_{k=0}^{\infty} |\alpha_{\nu} (Nm)^{-sk}| \leq d \prod_{\nu} \sum_{k=0}^{\infty} |\alpha_{\nu} (Nm)^{-sk}||Nm^{-s}|^{d}
\]

\[ \leq d \prod_{\nu} S_{\nu} p(s) \quad \text{where} \quad S_{\nu} = \text{the push Euler factor of} \ S_{\nu}(s)
\]

\[ \leq d S_{\nu}(s).
\]

The convergence now has been reduced to that of \( S_{\nu}(s) \),

which we take as a known \( \blacksquare \).

It is easy to see that if \( \nu \) is the trivial rep, then

\[ \mathcal{L}(s, p, L_{\nu}) = S_{\nu}(s). \]

Let \( p : \text{Gal}(L_{1}/K) \to GL(V) \) and \( p_{2} : \text{Gal}(L_{2}/K) \to GL(V_{2}) \)

be two representations. Define \( p_{1} \circ p_{2} \) to be the representation

from \( \text{Gal}(L_{1}/K) \to GL(V \otimes V_{2}) \). Then we have:

**Lemma 4.2:** \( \mathcal{L}(s, p_{1} \circ p_{2}, L_{\nu}) = \mathcal{L}(s, p, L_{\nu}) \mathcal{L}(s, p_{2}, L_{\nu}). \)
Lemma 4.3: Let $K \subseteq L \subseteq E$ be Galois extensions. Let $p : \text{Gal}(E/K) \to \text{GL}(V)$ be the rep.

Assume $p' : \text{Gal}(E/K) \to \text{GL}(V)$ is the rep.

Given by $p \circ \text{proj}$ where proj is the projection map

$$\text{proj} : \text{Gal}(E/K) \to \text{Gal}(^sE/K).$$

Then

$$L(s, p', E/L) = L(s, p', E/K).$$

Recall from last term the motion of an induced module:

Let $M$ be an $H$-module with $M \subseteq G$. Then we define a $G$-module

by

$$\text{Ind}_H^G M = \left\{ f : G \to M \mid f(gh) = h^{-1} f(g) \forall h \in H, g \in G \right\},$$

$G$-action given by $g \cdot f(x) = f(gx).$

Lemma 4.4: Let $K \subseteq L \subseteq E$ be Galois extensions, and set

$$H = \text{Gal}(E/L) \subseteq G = \text{Gal}(E/K).$$

Let $p : H \to \text{GL}(V)$ be a representation. Then $\text{Ind}_H^G p : G \to \text{GL}(V)$ is a representation and we have

$$L(s, p, E/L) = L(s, \text{Ind}_H^G p, E/K).$$

These three lemmas are all proved by using

character of representations and the fact that a rep. is determined by its character. The character of a rep.
\(\rho : G \to GL(V)\) is defined by

\[Xp : G \to C \]

\[Xp(g) = \text{tr}(p(g)).\]

We will not pursue this further here, but one can see Deme's book on the representation theory of finite groups for further information. To deal with infinite extensions one would use that the result for finite groups extend naturally to compact groups.

**Def.** Let \(\rho : G \to GL(V)\) be a representation. We say \(\rho\) is irreducible if \(V\) does not have a non-trivial \(G\)-invariant subspace.

**Artin's Conjecture.** Let \(\rho : GL(W) \to GL(V)\) be an irreducible representation. The Artin \(L\)-function \(L(s, \rho, W)\) defines an entire function.

We will now cover some review of the representation theory of finite groups. This will probably not be included in class.
We say two representations \((p, V)\) and \((p', V')\) are equivalent if the \(G\)-modules \(V, V'\) are isomorphic. We can factor each representation \((p, V)\) into a direct sum \(V = V_1 \oplus \cdots \oplus V_r\) of irreducible representations. Suppose \((p, V)\) is equivalent to \(r_\alpha\) of the representation \(V_i\), then we write

\[ p = \sum \alpha r_\alpha p_\alpha. \]

**Prop:** Two representations are equivalent if their characters are equal. If \(p = \sum \alpha r_\alpha p_\alpha\), then \(\chi_p = \sum \alpha r_\alpha \chi_{p_\alpha}\).

Let \(\chi_p\) be the character of the representation \(p : H \to GL(V)\).

The character of the representation \(p' = Ind_H^G p\) is given by

\[ \chi_{p'}(g) = \sum \chi_{p}(g_1 g_2 \cdots) \] where the \(g_i\) form an orbit of \(g_0 p(0)\) of \(G/H\).

**Thm. (Brauer's Thm.):** Every character \(\chi\) of a finite group \(G\) is a \(\mathbb{Z}\)-linear combination of characters \(\chi_i\) induced from characters \(\chi_i\) of degree 1 associated to subgroups \(H_i \triangleleft G\).

From this theorem we deduce the following result.
\[ S_L(n) = S_K(n) \prod_{x \neq 1} L(\sigma, x, L(1, \chi)) X(1) \]

where we have the right regular representation written as

\[ r_G = \sum_{x} \chi(x) x. \]

**Proof:** Let \( \chi \) be the trivial character of the subgroup \( \mu_1 \times \mathbb{C}^* \). The character \( \chi \) induces from this to \( \text{Gal}(\mathbb{C}(\mu_1)) \) in the right regular rep.

\[ r_G = \sum_{x} \chi(x) x. \]

We now just apply Lemma 9.9. \( \square \)
In this section we give mainly definitions and statements of results. We will see in subsequent sections that some of these results follow from more general results on ideal class characters and their $L$-functions.

Let $N$ be a positive integer. A Dirichlet character $\chi$ modulo $N$ is a group homomorphism

$$\chi : \left( \mathbb{Z}/N\mathbb{Z} \right)^* \to S^1 = \{ e^{2\pi i x} : |x| < 1 \}.$$  

We say the character $\chi$ is primitive if we cannot write $\chi$ as a composite

$$\left( \mathbb{Z}/N\mathbb{Z} \right)^* \to \left( \mathbb{Z}/M\mathbb{Z} \right)^* \to S^1$$

for some $M | N$ and Dirichlet character $\chi'$ modulo $M$. One can think of primitive characters as the building blocks of Dirichlet characters, much as primes are the building blocks of the integers.

If $\chi$ is primitive we say it is of conductor $N$. If $\chi$ is not primitive, the gcd of all $M$ so that $\chi$ can be written as a composite as above is the conductor of $\chi$. 
We extend \( x \) to a multiplicative function

\[ x : \mathbb{Z} \to S^1 \cup \{0\} \]

by

\[ x(m) = \begin{cases} x(m \text{ mod } N) & (m, N) = 1 \\ 0 & \text{otherwise} \end{cases} \]

If we consider the character modulo 1 defined by \( \chi(m) = e^{2\pi i m} \),

this is referred to as the principal character.

We define the Dirichlet \( L \)-function associated to \( x \) by

\[ L(s, x) = \sum_{m=1}^{\infty} \frac{x(m)}{m^s}. \]

(\( L(1, 1) = \delta(s) \).)

Prop 4.5: The series \( L(s, x) \) converges absolutely and uniformly

for \( \Re(s) > 1 + \delta \) for any \( \delta > 0 \). We also have

\[ L(s, x) = \prod_p (1 - x(p)p^{-s})^{-1}. \]

Proof: We prove this to the reader is familair with such

\([x(n)] \leq 1 \) for all \( n \in \mathbb{Z} \), and so if we write

\[ s = \sigma + it \] with \( \sigma > 1 + \delta \), then
\[
\left| \frac{1}{n^s} \right| = \frac{1}{n^s} \quad \text{and so}
\]

\[
\sum_{n=1}^{\infty} \left| \frac{X_n}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^s} < \sum_{n=1}^{\infty} \frac{1}{\eta^s}, \quad \text{which is}
\]

convergent. It only remains to prove the Euler product. Recall, this \( \prod p_a \) converges if

\[
\lim_{a \to \infty} a p_a \quad \text{is convergent, which is in turn equivalent to}
\]

\[
\sum_{n=1}^{\infty} \log p_a \quad \text{converging. Let} \quad E(s) = \prod_{p} \frac{1}{1 - p^{-s}}.
\]

Then

\[
\log E(s) = \sum_{p} \sum_{n=1}^{\infty} \frac{X(p^n)}{n^s} \quad (\text{Taylor expansion})
\]

This converges for \( s > 1 + \delta \) (check this). Thus,

the product \( E(s) \) must converge. Write

\[
\frac{1}{1 - X(p)^s} = 1 + \frac{X(p)^s}{p^s} + \frac{X(p)^{2s}}{p^{2s}} + \ldots \quad (\text{geometric series})
\]

Let \( p_1, p_2 \) be the primes less than \( M \). Then

we have

\[
\prod_{p \leq M} \frac{1}{1 - X(p)^s} = \sum_{n_1, \ldots, n_\infty} \frac{X(p_1^{n_1} \cdots p_\infty^{n_\infty})}{n_1^{s} \cdots n_\infty^s} = \sum_{n=1}^{\infty} \frac{X(n)}{n^s}
\]

where the \( i \) indicates the sum is only over \( n \) with \( n \)

only divisible by \( p_1, p_2 \). Write
\[ \prod_{p \leq M} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{n \leq M} \frac{\chi(n)}{n^s} + \sum_{n > M} \frac{\chi(n)}{n^s}, \] which makes sense since the sum necessarily contains all numbers less than \( M \) squared.

We then have

\[ \left| \prod_{p \leq M} \frac{1}{1 - \frac{\chi(p)}{p^s}} - L(\frac{\sigma}{2}, \chi) \right| \leq \left| \sum_{n \leq M} \frac{\chi(n)}{n^s} \right| \leq \sum_{n \leq M} \frac{1}{n^s}. \]

However, \( \sum_{n \leq M} \frac{1}{n^s} \to 0 \) as \( M \to \infty \) because this is the tail of a convergent series, and so

\[ \prod_{p \leq M} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{n = 1}^{\infty} \frac{\chi(n)}{n^s}. \]

Then \( \Re(s) > 1 + \varepsilon \), as claimed.

The Euler product essentially follows from the fundamental theorem of arithmetic, but it is good to see a proof at least once. We now want to give the functional equation for $L(s, \chi)$. This will relate $L(s, \chi)$ and $L(1-s, \bar{\chi})$ where $\overline{\chi(n)} = \chi(\overline{n})$. We give most of the details of the proof here as it is again important to see such things and to understand the framework for the use of ideal class characters.
The first step is to write \( L(s, x) \) as an integral.

Define \( z \in \mathbb{C}, \alpha > 0 \) by \( x(z) = (z)^\alpha \). Set

\[
\Gamma(x, s) = \Gamma\left(\frac{s + \varepsilon}{2}\right) = \int_0^\infty e^{-y} y^{s+\varepsilon} dy.
\]

This is the standard definition of the \( \Gamma \) function. We make the substitution \( y \rightarrow \frac{\pi n^2 y}{M} \) to obtain

\[
\left(\frac{M}{\pi}\right)^{\frac{s+\varepsilon}{2}} \Gamma(x, s) \frac{1}{n^s} = \int_0^\infty e^{-\pi n^2 y} y^{s+\varepsilon} \frac{dy}{y}.
\]

Multiply by \( x(n) \) and sum over all \( n \) to obtain

\[
\left(\frac{M}{\pi}\right)^{\frac{s+\varepsilon}{2}} \Gamma(x, s) L(s, x) = \sum_{n=1}^{\infty} x(n) \int_0^\infty e^{-\pi n^2 y} y^{s+\varepsilon} \frac{dy}{y}.
\]

We observe that

\[
\sum_{n=1}^{\infty} \left| \frac{\pi n^2 y}{M} \right| e^{-\pi n^2 y} y^{s+\varepsilon} \frac{dy}{y} \leq \left(\frac{M}{\pi}\right)^{\frac{s+\varepsilon}{2}} \Gamma\left(\frac{s+\varepsilon}{2}\right) \delta(t).
\]

Thus, we are free to interchange the integral and sum and obtain

\[
\left(\frac{M}{\pi}\right)^{\frac{s+\varepsilon}{2}} \Gamma(x, s) L(s, x) = \int_0^\infty \sum_{n=1}^{\infty} x(n) n^2 e^{-\pi n^2 y} y^{s+\varepsilon} \frac{dy}{y}.
\]
Let \( g(y) = \sum_{n=1}^{\infty} \frac{x(n)}{n^s} e^{-\pi n^2 y} \). The theta function associated to \( \chi \) is given by

\[
\theta(\tau, \chi) = S_1 + \sum_{n \in \mathbb{Z}} x(n) n^2 e^{\pi i n^2 \tau}.
\]

where

\[
S_1 = \begin{cases} 
1 & \text{if } \tau = 1 \\
0 & \text{otherwise}
\end{cases}
\]

From now on we include the case \( \tau = 1 \) purely for notational convenience. Thus, we should go through everything in this case (i.e., \( \mathcal{L}(\chi, 1) = \Delta(\chi) \)) and work on the details. Observe that we have

\[
\theta(z, \chi) = \sum_{n=1}^{\infty} x(n) n^2 e^{\pi i n^2 z/m}.
\]

since \( x(n) n^k = x(-n) (-n)^k \). Thus, we see that

\[
g(y) = \frac{1}{2} \theta(iy, \chi).
\]

Establishing such a relationship is always the key to proving functional equations.

**Def.** Let \( \chi \in \mathbb{Z} \) and \( \chi \) a primitive Dirichlet character modulo \( M \). The Gauss sum associated to \( \chi \) is
Proposition 4.6: For a primitive congruence, we have
\[ \tau(x, n) = \sum_{j=1}^{n-1} x(j) e^{2\pi j n / n} \]
and
\[ |\tau(x, n)| = \sqrt{n} \tau(x, 1) \]  

Proof: Exercise.

We state the following proposition without proof.

Proposition 4.7: Let \( x \) be prime, mod \( M \). Then
\[ \Theta(-\frac{1}{2}, x) = \frac{\tau(x, 1)}{\sqrt{L(x)}} \left( \frac{\pi}{x} \right)^{\frac{1}{2}} \Theta(2, x) \].

We can now use this result to deduce the functional equation for \( L(s, x) \). First, we must "complete the \( L \)-series", i.e., include \( L(s, 1/2) \). Let
\[ L_{\infty}(s, x) = \left( \frac{M}{x} \right)^{s/2} \Gamma(s) \Gamma(s - \frac{1}{2}) \].

The completed \( L \)-series associated to \( x \) is
\[ \Lambda(s, x) = \Gamma(\frac{1}{2}) L_{\infty}(s, x) \]
when \( \infty < \text{Re}(s) \leq 1 \), \( x \neq 1 \).
From this definition and the definition of \( \Theta(x) \), it is clear that we have

\[
\Lambda(s, x) = \frac{1}{2} \left( \frac{\pi}{\sqrt{m}} \right)^{3/2} \int_0^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y}.
\]

**Theorem 4.2:** Let \( x \) be a primitive character modulo \( M > 1 \). The completed \( L \)-series \( \Lambda(s, x) \) admits an analytic continuation to the entire complex plane and satisfies the functional equation

\[
\Lambda(s, x) = W(x) \Lambda(1-s, x)
\]

with

\[
W(x) = \frac{T(x)}{i^{s/2} \sqrt{M}}.
\]

where we write \( T(x) = T(\gamma, x) \).

**Proof:** We begin by splitting the integral defining \( \Lambda(s, x) \) into 2 pieces (set \( s' = \frac{s+\epsilon}{2} \))

\[
\Lambda(s, x) = \frac{1}{2} \left( \frac{\pi}{\sqrt{m}} \right)^{3/2} \left[ \int_0^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} + \int_1^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} \right].
\]

\[
= \frac{1}{2} \left( \frac{\pi}{\sqrt{m}} \right)^{3/2} \left[ \int_0^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} + \int_1^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} \right] \quad \left( \text{by integral} \right)
\]

\[
= \frac{1}{2} \left( \frac{\pi}{\sqrt{m}} \right)^{3/2} \left[ \int_0^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} + \int_1^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} \right]
\]

\[
= \frac{1}{2} \left( \frac{\pi}{\sqrt{m}} \right)^{3/2} \left[ \int_0^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} + \int_1^\infty \Theta(iy, x) y^{s+\frac{1}{2}} \frac{dy}{y} \right] \quad \left( \text{Prop 4.7} \right)
\]
It follows from the integral representation that $\Lambda(s, \chi)$

takes analytic continuation to all of $\mathbb{C}$. This is essentially
due to the fact that $\Theta(\imath y, \bar{\chi})$ and $\Theta(\imath y, \chi)$ are

$O(e^{\pi y/\text{Im} \chi})$ where one says $f(y) = O(g(y))$ when $f$

a bounded function $\forall y < \epsilon$ with $f(y) = O(\sqrt{y} \log y)$ as $y \to \infty$.

One should be sure to work out the details of this convergence.

As above, we calculate

$$\Lambda(1-s, \chi) = \frac{1}{2} \left( \sum_{M \leq A} \frac{\chi(M)}{M} \right) \left[ \int_{1}^{\infty} W(x) \Theta(\imath y, \chi) y^{-s} dy + \int_{1}^{\infty} \Theta(\imath y, \bar{\chi}) y^{-s} dy \right].$$

As an exercise, one should prove that this integral

converges for all $s \in \mathbb{C}$ and we obtain

$$\Lambda(s, \chi) = W(x) \Lambda(1-s, \chi).$$

One should note that $|W(x)| \leq \sqrt{x}$ by Prop 9.6.

Though we will later see how Dirichlet characters are examples

of Hecke characters and ideals class characters and so we will obtain the

relationship with Euler $L$-functions this way, in the case we can

easily obtain the relationship directly.
Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character.

Let $\mathcal{S}_N$ be a primitive $N^2$-root of unity. From Shale's theory we have an isomorphism

$$\left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)^\times \xrightarrow{\varphi} \text{Gal}\left(\mathbb{Q}(\mathcal{S}_N)/\mathbb{Q}\right)$$

$$m \mapsto (\mathcal{S}_N \mapsto \mathcal{S}_N^{-m}).$$

This isomorphism gives a bijection between Dirichlet characters and complex one-dimensional representations of $\text{Gal}(\mathbb{Q}(\mathcal{S}_N)/\mathbb{Q})$.

Namely, given $\chi$ we obtain $\rho_\chi : \text{Gal}(\mathbb{Q}(\mathcal{S}_N)/\mathbb{Q}) \to GL_1(\mathbb{C})$

given by $\rho_\chi(\sigma) = \chi(\sigma) \cdot \mathcal{S}_N^{\sigma^{-1}(\sigma)}$. This acts on $\mathcal{S}_N$ via

$$\rho_\chi(\sigma) \cdot \mathcal{S}_N = \chi(\sigma) \cdot \mathcal{S}_N^{\sigma^{-1}(\sigma)}.$$

On the other hand, given $\rho : \text{Gal}(\mathbb{Q}(\mathcal{S}_N)/\mathbb{Q}) \to GL_1(\mathbb{C})$, we define

$$\chi_\rho : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \text{ in } \chi_\rho(m) = \rho(\mathcal{S}_N^m).$$

It remains to check that the $L$-functions match up. Extend $\chi$ to all of $\mathbb{Z}$. The $L$-function is given by

$$L(s, \chi) = \prod_p \left(1 - \chi(p) \cdot p^{-s}\right)^{-1}$$

$$= \prod_{p \mid N} \left(1 - \chi(p) \cdot p^{-s}\right)^{-1}.$$
When we have used that \( \chi(p) = 0 \) \( \forall p \nmid N \). Let \( p \mid N \). We know that under \( \wp \) we have \( p \mapsto \wp(p) = \operatorname{Frob}_p \). Thus, 
\[
\beta(x(\wp^*(\operatorname{Frob}_p))) = \chi(p).
\]
Thus, for \( p \mid N \), we have
\[
L_p(s, \chi) \cdot (1 - \beta(x(\operatorname{Frob}_p)) \cdot p^{-s})
= 1 - \chi(p) \cdot p^{-s}
= L_p(s, \chi).
\]
Now consider \( p \nmid N \). In this case \( p \) is ramified in \( \mathbb{Q} \) and \( p = \mathcal{I}_p \cdot \mathfrak{f}_p \). Since \( \beta(x(\operatorname{Frob}_p)) \) acts via multiplication on \( \mathbb{C} \) we have that \( \mathcal{I}_p \cdot \mathfrak{f}_p = \mathfrak{f}_p \cdot \mathcal{I}_p = \mathfrak{f}_p \) (nothing but 0 can be prime by mult. by something with the 2). Thus, in this case we see that \( \beta(x(\operatorname{Frob}_p)) \big|_{\mathcal{I}_p} = 0 \) and so \( L_p(s, \chi) = 1 \). Thus, we have
\[
L(s, \chi) = \prod_p (1 - \beta(x(\operatorname{Frob}_p)) \cdot p^{-s})^{-1}
= \prod_{p \nmid N} (1 - \chi(p) \cdot p^{-s})^{-1}
= L(s, \chi),
\]
as desired.
In this section we see how one can generalize the notion of Dirichlet characters. First, we shed some light on how to properly view Dirichlet characters to see the generalization. Recall from the Kronecker-Weber theorem that any finite abelian extension of \(\mathbb{Q}\) sits inside \(\mathbb{Q}(\zeta_N)\) for some \(N \geq 1\). We saw this by observing that any modulus \(m\) of \(\mathbb{Q}\) must divide \(\infty \cdot N\) for some positive integer \(N\) and then using that the ray class field of \(\infty \cdot N\) is precisely \(\mathbb{Q}(\zeta_N)\). The ray class group \(\mathfrak{c}\) of \(\infty \cdot N\) is given by \((\mathbb{Z}/\mathfrak{c})^*\), as you calculated in an exercise. Thus, we can view Dirichlet characters as character of the ray class groups of \(\mathbb{Z}\) arising from the modulus \(\infty \cdot N\) for \(N \geq 1\). This is the viewpoint that leads to the most straightforward generalization to other fields.
Let \( K \) be a number field and \( m \) a modulus of \( K \).

Let \( C_K^m \) be the Ray class group associated to the modulus \( m \),
\( K^m \) the associated Ray class field, and \( \omega \) the conductor of \( K^m \).

We can define a Dirichlet character of conductor \( \omega \) associated to \( K \) to be a group homomorphism
\[
\chi : C_K^m \to \mathbb{C}.
\]

As was mentioned above, this is clearly a generalization of the Dirichlet characters of \( \mathbb{Q} \) considered in the previous section.

Though we are focused on characters this section, we briefly mention how one can define the L-function associated to such a \( \chi \). Recall that \( C_K^m = \mathbb{Z}^m / \mathbb{Z} \omega \).

Write \( \mathfrak{a} \) for an ideal \( \mathfrak{a} \in I^m \) when regarded in \( C_K^m \).

\( \mathfrak{c} \) is an ideal of \( \mathcal{O}_K \) and in \( I^m \). Set \( X(\mathfrak{c}) = 0 \).

Then we define
\[ L(s, \chi) = \sum_{\chi(\alpha) = 1} \frac{\chi(\alpha)}{\alpha^s} \]  

when we write \( \chi(\alpha) \) for \( \chi(\alpha) \) when \( \alpha \in \mathbb{Z}^* \). We will see this is a special case of more general \( L \)-functions later, so we do not pursue the properties here.

We move now to the adelic setting. Recall that when stating the main result of \( \text{GL}_1 \), it was more natural to state the results in terms of ideals, and then transfer these results to statements in terms of ideals. It is much the same here and we work with ideal class characters.

**Def.** A continuous group homomorphism

\[ \chi: \mathbb{A}^\times \mathbb{F}_K \rightarrow \mathbb{C}^\times \]

is called an \textit{ideal class character}. We say the character is \textit{unitary} if \( |\chi(\alpha)| = 1 \) for all \( \alpha \in \mathbb{A}^\times \mathbb{F}_K \).
One should note that in many texts authors refer to an "character" as "quasi-character" and an "unitary character" as "character".

**Def.:** We say an idele class character is unramified at \( v \in M_v \) if \( \chi(v) = 1 \), where \( \chi_v = \chi|_{\mathbb{A}_v} \) and \( \mathbb{A}_v = \mathbb{A}_F^v \).

Recalling that \( \mathbb{A}_v \) corresponds to the ideles at \( v \) under the local Artin map, this definition makes sense in terms of the usual meaning of unramified twisting on the idele group.

A group is called a totally disconnected locally compact (t.d.l.c.) group if it is a topological group having a basis of neighborhoods of 1 consisting of open and compact subgroups.

**Examples:** Let \( F_v \) be a non-archimedean local field. Then \( F_v, F_v^x, \) and \( \text{GL}_n(F_v) \) are all totally disconnected locally compact groups.
Lemma 4.9: Let $G$ be a totally disconnected locally compact group.

Let $X$ be a continuous character

$$X : G \rightarrow \mathbb{C}^*.$$ 

Then $\ker X$ must contain an open subgroup $H$.

**Proof:** Exercise. Let $N \subset \mathbb{C}^*$ be a nbhd of $1$ small enough so it contains no nontrivial subgroup of $\mathbb{C}^*$. Then $X(N)$ is a nbhd of $1$ and so contains $H$, a compact open subgroup containing $1$. However, $\phi(H) \subset N$ and so $\phi(H) \neq 1$ in $G$, $H \subset \ker X$. \(\square\)

Let $\mathbb{A}_k^\infty$ denote the finite adele, i.e.,

$$\mathbb{A}_k^\infty = \left\{ x \in \mathbb{A}_k : \prod_{v \in \mathbb{Q}_\infty} x_v \in \mathbb{R}_+^* \cap \mathbb{Q}_\infty^* \right\}. $$

Note that $\mathbb{A}_k^\infty$ is a totally disconnected locally compact group because it is the product of totally disconnected locally compact groups.

Prop 4.10: Let $X$ be a character of $\mathbb{A}_k^\infty$. Then there exists a finite set of places $S \subset M_k$, with $M_\infty \not\subset S$, s.t. $X$ is unramified at all $v \not\in S$. 
Proposition: Consider \( \chi \) restricted to \( \mathbb{A}_K^\times \), call this character \( \chi_f \). Clearly \( \chi_f \) is continuous and so by Lemma 4.9 the kernel of \( \chi_f \) contains an open neighborhood of 1. We know any open neighborhood of 1 must contain \( \bigcap_{v' \in S'} \mathcal{O}_v^\times \) for some finite set \( S' \) by the definition of the topology of \( \mathbb{A}_K^\times \). Thus, there is some \( \mathcal{O}_v \) with 1 in it to obtain the desired \( S' \).

Suppose that \( \chi \) is unramified at \( v \). Since \( \chi_v(x^v) = 1 \), we see that \( \chi_v \) is well-defined.

Definition: We say a character \( \chi: \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times \) is of \emph{finite order} if \( \exists m \in \mathbb{Z}^\times \) such that \( \chi(y)^m = 1 \) for all \( y \in \mathbb{A}_K^\times \).

We now spend some time on more abstract results so that we do not have to repeat things in different settings.

Definition: A topological group \( G \) is called \emph{spherically compact} if it is the direct product of a commutative compact group \( H \), and \( G/H \) isomorphic to either \( \mathbb{IR} \) or \( \mathbb{IZ} \).

Example: 1. Let \( K_v \) be a non-archimedean local field. Then we have

\[ \chi_v(x^v) = 1 \]
\[ K_v^* = \mathbb{C}^* \times \mathbb{C}^{\mathbb{N}} = \mathbb{C}^* \times \mathbb{C}. \]

Since \( \mathbb{C}^* \) is compact, we have that \( K_v^* \) is quasi-compact.

Consider the idele class group \( \mathbb{A}^*_k/k \). Recall the norm map

\[ 1 \times 1 : \mathbb{A}^*_k \to \mathbb{R}_{\geq 0}. \]

We saw before that if we set \( (\mathbb{A}^*_k)^2 \) to be the kernel of this map, then \( (\mathbb{A}^*_k)^2 \) is compact. We find we have an exact sequence

\[ 1 \to (\mathbb{A}^*_k)^2 \to \mathbb{A}^*_k \to \mathbb{R}_{\geq 0} \to 1. \]

Since \( \mathbb{R}_{\geq 0} = \mathbb{R} \) as a group under \( \times \), we have that \( (\mathbb{A}^*_k)^2 \) is quasi-compact.

**Def:** Let \( G \) be quasi-compact and \( \chi \) a character of \( G \). We say [

\( \chi \) is a principal character of \( \chi \).]

The set of characters of a quasi-compact group \( G \) clearly form a group under \( \times \), and we write \( \hat{X}(G) \) for the group of characters. We write \( \hat{X}(G)_1 \) for the group of principal characters.
Lemma 4.11: Let \( X \) be a character of a compact group \( G \). Then \( X \) is unitary.

Proof: The fact that \( G \) is compact implies that \( g \mapsto |X(g)| \)
must map into a compact subgroup of \( \mathbb{R}_{\geq 0} \). However, the only such group is \( \{1\} \). Thus, \( |X(g)| = 1 \forall g \in G \). \( \square \)

Prop 4.12: Let \( G \) be a quasi-compact group. There is a continuous character of \( G \) into \( \mathbb{R}_{\geq 0} \). By \( X \) in such a character, then the kernel of \( X \) is \( G_1 \). Every character of \( G \) into \( \mathbb{R}_{\geq 0} \)
can be written uniquely in the form
\[ g \mapsto X(g^a) \]
for some \( a \in \mathbb{R} \).

Proof: Write \( G = G_1 \times N \) where \( N \not\subset \mathbb{R} \). Lemma 4.11 implies
that every character of \( G \) into \( \mathbb{R}_{\geq 0} \) must be trivial when restricted to \( G_1 \). Thus, if we write \( g = (g_1, n) \in G = G_1 \times N \),
we see \( X \) must be of the form
\[ X(g) = X(g_1, n) = \phi(n) \]
where \( \phi \) is a character of \( N \) into \( \mathbb{R}_{\geq 0} \).

First suppose \( N \not\subset \mathbb{R} \). In this case, we see \( \phi \) is a homomorphism.
from \( \mathbb{R} \) into \( \mathbb{R}^n \). Thus, we can write \( \phi_n \circ \log \phi_n \) as a homomorphism \( \mathbb{R} \to \mathbb{R}^n \) with the action being addition. However, any such map \( \alpha \) is of the form \( n \mapsto \alpha_n \) for some \( \alpha \in \mathbb{R} \).

Thus, we can write \( \phi(n) = \exp \alpha \).

Now suppose \( N \geq 2 \). In this case we have that \( \phi(n) = b^n \) for some \( b \in \mathbb{R} \). Thus, we can write \( \phi(n) = \exp \alpha \) for \( \alpha = \log b \).

Thus, in both cases we see \( \phi \) is a homomorphism of \( \mathbb{R} \). This shows the existence of the mentioned character \( \chi \).

Then, such a \( \chi \), we can write \( X(g, n) = \exp \alpha \).

for some \( \alpha \in \mathbb{R} \). It is clear that this has kernel \( G_\alpha \). Moreover, given another character \( \psi \), we can find a \( \alpha \) such that \( \psi = \exp \alpha \).

Letting \( \sigma = \chi \circ \psi \), we have \( \sigma = \chi \circ \psi = \chi \circ \exp \alpha = \exp (\alpha + \sigma) \).

is uniquely determined.

**Corollary 4.13:** Let \( G, G_\alpha \), and \( X \) be as in the previous prop. Then

\[ X(g) = \begin{cases} \mathbb{C} & \text{if } G/\alpha = \mathbb{R} \\ G & \text{if } G/\alpha = \mathbb{Z} \end{cases} \]

Every character \( \chi \) is of the form

\[ \chi \circ (g) = \chi(g)^3 \]

with \( s \in \mathbb{C} \). The map \( s \mapsto \chi \) is a morphism of \( \mathbb{C} \) onto
\[ X(G)_2 \text{ with } \]
\[ \chi \oplus \psi = \begin{cases} \chi & \text{if } \frac{\chi}{\psi} = \mathbb{Z} \\ \chi \oplus \psi & \text{if } \frac{\chi}{\psi} = \mathbb{Z}_p \end{cases} \]

**Proof:** Let \( \omega \) be any character of \( G \). Consider the character

\[ \begin{align*}
G & \rightarrow \mathbb{R} \\
G & \rightarrow \mathbb{R}_{>0}.
\end{align*} \]

Then, Prop. 4.12 shows \( \exists \sigma \in \mathbb{R} \) s.t. \( \sigma(x) = x(x)^\sigma \). Let \( \sigma \in \mathbb{R} \). The map \( g \mapsto \psi_0^{-1}(g) \psi(g) \) is then a unitary character of \( G \). Call this unitary character \( \psi_0 \).

Anyone may that \( \psi(x) \chi(x) = \psi_0(x) \). Then clearly we have \( \psi \) is also trivial on \( G \), and we can write \( \psi(x) = \psi_0(x) \). As in the proof of Prop. 4.12, choose \( a \in \mathbb{R} \) s.t. \( \chi(x) = \exp(a, x) \).

**Prop:** \( N \mathbb{R} \). Then it is a classical result that any unitary character of \( \mathbb{R} \) can be written as

\[ \begin{align*}
N & \rightarrow \exp(\mathfrak{t}N) \\
N & \rightarrow \exp(\mathfrak{t}(N))
\end{align*} \]

for some \( \mathfrak{t} \in \mathfrak{t} \). Write \( \psi(n) = \exp(n \mathfrak{t}) \). Thus, we have

\[ \psi = \psi_0 \] with \( \psi = \sigma + \mathfrak{t}i \). Similarly, if \( N = \mathbb{Z} \), we can write \( \psi(x) = \exp(x \mathfrak{t}) \) and the same as above. Now \( \mathfrak{t} \) is uniquely determined by \( \sigma \) if \( N = \mathbb{R} \). But \( N = \mathbb{Z} \), \( \mathfrak{t} \) is uniquely determined modulo \( \mathbb{Z} \) if \( N = \mathbb{Z} \). Thus, we see...
The map \( s \rightarrow w \), is an isomorphism of \( G \) onto \( X(G)_2 \) if \( N \subseteq R \).

If \( N = \mathbb{Z} \), then \( w(g, n) = u^n \), with \( u = \exp(2\pi i) \) and \( u \rightarrow \infty \) is an isomorphism of \( G \) onto \( X(G)_2 \).

**Corollary 4.14:** Let \( G \) be quasi-compact. The group \( X(G) \) of characters of \( G \) is isomorphic to the direct product of \( X(G)_1 \) and the group of unitary characters of \( G \), trivial on \( N \). This latter group is isomorphic to the dual of \( G \).

**Proof:** Let \( w \in X(G) \). We can in the previous proof that we could uniquely write \( w = w' w' \), where \( w' \) is a unitary character of \( G \) and \( w' \in \mathbb{R} \). Since \( w' \) is a character of \( G \), it can be written as \( w'_1 w'_2 \) with \( w'_1 \) trivial on \( G \), and \( w'_2 \) trivial on \( N \). Then

\[
\tilde{w} = (w_0 w'_1) w'_2 \quad \text{and} \quad \tilde{w} \in X(G)_1.
\]

**Lemma 4.15:** Let \( G \) be L.D.C. Then every character of \( G \) into \( C^* \)

is locally constant.

**Proof:** Extensive.

**Prop 4.16:** Let \( K \) be a nonarchimedean field, \( char(K) = p \). The principal characters of \( K^* \) are those of the form

\[
x \mapsto (x)^p.
\]
and $x \in G$. The group $X(k^\times)$ is a direct product of $X(k^\times)^2$ and the group of unitary characters $\Phi(k^\times)$ defined by $\Phi(x) = 1$.

**Prop:** Exercise.

**Prop 4.17:**

1. Every character of $R^\times$ can be written uniquely as
   $$x \mapsto \text{sgn}(x)x^s$$
   with $\text{sgn}(x) = \frac{x}{|x|},$ $x \in R,$ and $s \in C.$

2. Every character of $C^\times$ can be written uniquely as
   $$x \mapsto e^{i\theta(x)}$$
   for $x \in C.$ $\theta \in \mathbb{Z}.$

**Proof:**

1. For $R^\times$, we have $R^\times \cong G \times N$ with $G = \{ \pm 1 \}$ given by
   $$R^\times \to G \times R^\times \to G \times R \times R$$
   $$x \mapsto (\text{sgn}(x), |x|) \to (\text{sgn}(x), |x|^2).$$
   Observe that we can choose $x$ in Prop 4.12 to be $x \mapsto |x|$. Thus, all the principal characters can be written uniquely as $x \mapsto |x|^{2s}$ for some $s \in C$. The characters of $G$ and $\text{sgn}(x) \mapsto \text{sgn}(x)$ of $\text{sgn}(x) \mapsto x,$ which combined with Cont. 4.12 yields the result.

2. For $C^\times$, we have $C^\times \cong G \times N$ with $G = S^1$ and $N \cong R^\times$ given by
   $$C^\times \to S^1 \times R^\times \to S^1 \times R$$
   $$e^{i\theta(x)} \mapsto (e^{i\theta(x)}, x) \mapsto (e^{i\theta(x)}, \log|x|).$$
At is a classical result that the characters of $S^1$ are of the form $e^{i\theta} \mapsto e^{i\theta n}$ for $n \in \mathbb{Z}$ and we have already seen we can write the character of $\mathbb{R}^\times \mapsto x \mapsto 1x^5, \ldots$.

Thus, we have characters in $X(\mathbb{C}^\times)$ are of the form

$$Z = re^{i\theta} \mapsto r^n e^{i\theta n}.$$ 

Finally, the last main application of Prop 4.12 we are interested in is to the group $\mathbb{M}_k^\times$. Recall in this case we have

$$\left( \mathbb{A}_k^\times \right)^2 / \mathbb{M}_k^\times \rightarrow \mathbb{M}_k^\times / \mathbb{M}_k^\times \mapsto \mathbb{P} \rightarrow \mathbb{1}.$$

From this we have the following, which follows exactly as before.

Prop 4.18: Every character in $\mathbb{M}_k^\times / \mathbb{M}_k^\times$ can be written in the form $w(x) = \mu(x) 1x^5$ for a unique $S \in \mathbb{C}$ and \mu a unitary character.

We can now use what we have done to show that

Dirichlet characters are just special instances of unitary idele class characters.
Prop 4.19: Let \( K \) be a number field and let \( \chi \) be a unitary character of finite order. Then there exists a modulus \( m \) s.t.

\[ m =\prod_{\nu \nmid m} m_\nu \]

so that \( \chi|_{K_m} = 1 \) when \( m \) recall

\[ W_K^m = \prod_{\nu \nmid m} W_K^m(\nu) \times \prod_{\nu \nmid m} K_\nu^\ast \times \prod_{\nu \nmid m} J_\nu^\ast \]

\[ W_K^m(\nu) = 1 + \mathfrak{a}_\nu^{m_\nu} \]

We also recall the definition of \( \mathcal{A}_m^{K} \):

\[ \mathcal{A}_m^{K} = \prod_{\nu \nmid m} W_K^m(\nu) \times \prod_{\nu \nmid m} K_\nu^\ast \]

with the usual restricted direct product. Prop. 4.14 showed that

\[ \mathcal{A}_m^{K}/K_m, W_K^m \xrightarrow{\phi} \mathcal{A}_m^{K}/K_m, W_K^m \xrightarrow{\phi} C_m^\ast \]

Define \( \chi_0 \) to be the character defined by \( \chi_0 = \chi \cdot \phi^m \).

The fact that \( \phi^m(\chi_0(\nu)) = \chi(\nu) \) for \( \nu \nmid m(\nu) \) follows.
immediately from the definition of $\Gamma$. Thus, we see how
to associate a Dirichlet character to $\pi$. Now suppose we
have a Dirichlet character $\chi_0$ modulo $M$. Company with
$\psi$ gives a unitary idele class character, which has finite
order $b/c$ to $\phi$ also. Thus, we only need to require that when
we go from $\pi$ to $\chi_0$ that the $M$ be minimal to ensure this
is a bijection and the character are primitive.
In this brief section, we introduce the $L$-functions attached to an idele class character $\chi$ and state the main results.

Let $K$ be a field and $\chi$ an idele class character. As before, we write $\chi = \mu \cdot \chi^*$ for $\mu$ a unitary character and $\chi^* \in C$. We define $L_v$ as before, namely,

$$L_v : K_v \to \mathbb{C}^\times$$

$$L_v(t) = \chi \left( \frac{1}{t}, \ldots, \frac{1}{t}, 1, \ldots \right).$$

Let $v$ be a finite place. We define the local $L$-factor at $v$ by

$$L_v(\chi_v) = \begin{cases} (1 - \chi_v(p_v))^{-1} & v \text{ uniform}, \\ 1 & v \text{ non-uniform}. \end{cases}$$

Recall that since $\chi_v(\mathbb{A}^c) = 1$ for $v$ uniform, $\chi_v(c_0) = \chi(p_v)$.

Define

$$L(\chi) = \prod_{v \in \mathcal{S}} L_v(\chi_v).$$

**Prop. 4.20:** Let $\chi = \mu \cdot \chi^*$ as above. Then $L(\chi)$ is absolutely convergent whenever $\Re(\chi) > 1$. 
Proof: We have already seen that $z$ is ramified at most finitely many places. The local $L$-factor for these places are all 1, so we can ignore them.

Writing $q_v = |\mathfrak{o}_v|$, we have

$$\prod_v L_v(\chi, 1) = \prod_v \left( \frac{1}{1 - \chi_v(q_v)|\mathfrak{q}_v|} \right).$$

As before, an infinite product converges iff the logarithm of the product converges. Thus, we want to show

$$\log \left( \prod_v L_v(\chi, 1) \right)$$

converges for $\Re(z) > 1$. We have

$$\log \left( \prod_v L_v(\chi, 1) \right) = \sum_v \log \left( \frac{1}{1 - \chi_v(q_v)|\mathfrak{q}_v|} \right)$$

$$= \text{Re} \left( \sum_v \log \left( \frac{1}{1 - \chi_v(q_v)|\mathfrak{q}_v|} \right) \right)$$

$$= \text{Re} \left( \sum_v \sum_{n \geq 1} \frac{\chi_v(n)|\mathfrak{q}_v|^{-n}}{n} \right)$$

when we have used $\log z = \log |z| + i \arg(z)$.

We may use that $\text{Re}$ is additive to conclude that it is enough to establish the convergence of the series.
\[
\sum \sum \frac{1}{m_0^{-1} m_g^{-m_g}}
\]

where \( \tau \in \mathbb{R}_+ \). We can rewrite this as

\[
\sum \frac{1}{\nu \lambda_0 \nu_0} \frac{1}{m_0^{-1} m_g^{-m_g}}
\]

We know that \( \nu = \nu_{\lambda_0} \leq n = (K \cdot \delta) \). For \( \nu \lambda_0 \), we have that \( g_0 \) is an integer power of \( \rho \), and so the series is bounded by

\[
\sum \frac{1}{\nu \lambda_0 \nu_0} \frac{1}{m_0^{-1} m_g^{-m_g}}
\]

\[= n \log \left( \prod \frac{1}{1 - \rho^{\lambda_0}} \right) \quad \text{(using Taylor series again)}
\]

However, for \( \sigma > 1 \) we have already seen that \( \prod \frac{1}{1 - \rho^{\lambda_0}} \)

converges, and so we are done. \( \blacksquare \)

**Def.** Let \( \chi \) be an idèle class character for the number field \( K \).

Define the **Hecke-Hom** associated to \( \chi \) by

\[
L(s, \chi) = L(s, \chi \cdot 1)^{\sigma} \quad \text{where}
\]

\[
L(s, \chi) = \prod_{\nu \in \mathcal{O}_K} \frac{1}{1 - \chi(\nu) \tau_s \nu}
\]
Exercise: Show that the Hecke $L$-functions generalize the

Riemann zeta function, Dedekind zeta function, and

the Dirichlet $L$-functions (including those for $K/\mathbb{Q}$).

In order to state the meromorphic continuation and functional

equation for the Hecke $L$-functions, we must include the $L_v$, $v$ for $v \neq \infty$, as was the case for Dirichlet characters. First suppose

that $v \neq \infty$ and $v$ is a real place. Since we are assuming

that $v$ is a real place, we must have

$$
\chi_v(x) = \delta_{v}(x) \mathbf{1}_v(x)
$$

for $x \in \mathbf{C}_v$, $x \in \mathbb{R}$. Define

$$
L_v(s) = \pi^{-\frac{\delta_v}{2}} \Gamma\left(\frac{\delta_v}{2}\right)
$$

Now suppose $v$ is a complex place. Since that $v$ is complex place,

$$
\chi_v(x) = r \chi_v(x)
$$

for $x \in \mathbf{C}_v$, $x \in \mathbb{C}$. Define

$$
L_v(s) = (2\pi)^{-\frac{\delta_v}{2}} \Gamma\left(\frac{\delta_v}{2}\right)
$$

We will see in subsequent sections how these factors are
from. We write
\[
\Lambda(s,x) = \left( \prod_{\nu \mid x} \Lambda_s(\nu \cdot x) \right) \Lambda(s,x), \quad \text{where} \quad \Lambda_s(\nu \cdot x) = L_s(\nu \cdot x | x^{1/2})
\]

We have the following theorem, the proof of which will occupy the next couple of sections, and is the main result of Tate's thesis.

**Theorem 1.8:** Let \( \chi \) be a unitary ideal class character. The function \( \Lambda(s,x) \) admits a meromorphic continuation to \( \mathbb{C} \) and satisfies the functional equation
\[
\Lambda(s,x) = \varepsilon(s,x) \Lambda(1-s,x)
\]

where
\[
\varepsilon(s,x) = \prod_{\nu \mid x} \varepsilon(\nu \cdot x | x^{1/2}) \in \mathbb{C},
\]

(We will define \( \varepsilon(s,x) \) explicitly in subsequent sections.)

We will prove the theorem by an analysis of certain zeta integrals. It is the same field as what we did for Dirichlet characters, so one should keep that elementary case in the back of one's mind. One should also be careful not to confuse the
"\( s_0 \) in \( X \times \mathbb{R}_{\geq 0} \) and the "s" in the definition of \( L(s, \chi) \).

The "s" is fixed, depending on \( \chi \). The "s" is the variable for our \( L \)-function.
§ 4.6 Local Zeta Integers

Let $K$ be a local field with char($K$) = 0, and let $\chi_0$ be a character of $K^\times$. We retain the definitions and notations for $L_s(\chi_0)$ and $L(2, \chi_0)$ from the previous section. We define the shifted dual of $\chi_0$ by $\tilde{\chi}_0 = \chi_0^{-1} \cdot 1_{K^\times}$. This is defined so that

$$L_s \left( \left( \chi_0, (\tilde{\chi}_0)^{-1} \right) \right) = L(2-s, \chi_0).$$

We drop the subscripts on $\nu$ from our notation for this section for convenience.

**Def.** The dual group of $K$ is $\text{Hom}(K, S^1)$, where $K$ is thought of as an abelian group. We write $\hat{K}$ for the dual group.

Let $\Psi \in \hat{K}$ be nontrivial. One can show that if $\phi \in \hat{K}$, then $\exists x \in K$ such that $\phi(x) = \Psi(ax)$ for all $x \in K$. We write $\Psi_a$ for the map $x \mapsto \Psi(ax)$. Thus, we have

$$K \rightarrow \hat{K}$$

$$a \mapsto \Psi_a$$

so an isomorphism of topological groups.
Let $G$ be any locally compact abelian group with bi-invariant Haar measure $d\nu$. (Recall a Haar measure is left-invariant if $\mu(xA) = \mu(A)$ for $A \subseteq G$. Similarly for right-invariant. The measure is bi-invariant if it is left and right invariant.)

**Def:** Let $f \in L^1(G)$. The Fourier transform of $f$ is the function $\hat{f} : \mathbb{C} \to \mathbb{C}$ defined by

$$
\hat{f}(x) = \int_G f(y) \overline{\chi(y)} \, d\nu(y)
$$

for $x \in \mathbb{C}$.

The main result we will need is the Fourier inversion formula: Before we state this we need the following definition.

**Def:** A Haar measurable function $\phi : G \to \mathbb{C}$ is said to be of *positive type* if for any $f \in L^2(G)$, the following inequality holds:

$$
\int_G \int_G \phi(s^{-1}t) f(s) \overline{f(t)} \, d\nu(s) \, d\nu(t) \geq 0.
$$
Recall the definition of $L^\infty(\mathbb{R})$: Let $f: \mathbb{R} \to [0,\infty)$ be a measurable function. Let

$$S = \{ a \in \mathbb{R} : \mu(f^{-1}(a,\infty)) = 0 \}$$

The essential sup. of $f$ is defined by

$$\text{ess sup } f = \begin{cases} \inf S & \text{if } S \neq \emptyset \\ \infty & \text{if } S = \emptyset \end{cases}$$

Let $\text{ess sup } f$ be the essential sup of $f$. Then

$$L^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to [0,\infty) : \text{ess sup } f \text{ is finite} \}$$

These are essentially bounded functions.

Let $V(G)$ be the complex span of cont. fn. of positive type on $G$. Then

$$V'(G) = V(G) \cap L^1(G).$$

**Theorem 4.23:** There exists a Haar measure $dx$ on $G$ s.t. for all $f \in V'(G)$,

$$f(y) = \int_G \hat{f}(x) x(y) \, dx.$$

Moreover, we may identify $V'(G)$ with $V'(G)$. 
The measure $dx$ given in the Fourier inversion theorem is referred to as the dual measure of $dx$. In our current situation of interest, namely $C := \mathbb{R}$, we have that $\mathcal{K} = \mathbb{R}$ and so we say the measure $dx$ is self-dual if $dx = dx$. We also define a multiplication measure on $\mathcal{K}$ by $d^\times x = c \frac{dx}{|x|}$ for some fixed $c \in \mathbb{R}_{>0}$. (If $K$ is not, we always take $c=1$.)

**Defn:** Let $f$ be a complex-valued function on $K$. We say $f$ is smooth if

$$f \text{ is smooth if }$$

$$\begin{cases} f \text{ is } C^\infty & K \text{ arch.} \quad \text{and} \\ f \text{ is locally constant } & K \text{ arch.} \end{cases}$$

We recall that

$$p_{x_1} f_{x_1} \rightarrow 0 \text{ as } x \rightarrow \infty$$

for all polynomials $p(x)$. A **Schwartz–Berezin function** is a Schwartz function if $K$ is arch and a locally constant function of compact support if $K$ is monad.

We denote the space of Schwartz–Berezin functions by $S(K)$. 

Let \( f \in S(\mathbb{R}) \) and let \( \Phi \in \mathbb{C} \). Under the transform
\[
y \mapsto \hat{y} : \mathbb{R} \to \mathbb{R},
\]
we see that the Fourier transform
\[
\hat{f}(y) = \int_{\mathbb{K}} f(x) \overline{\Phi(xy)} \, dx = \int_{\mathbb{K}} f(x) \Phi(xy) \, dx.
\]
For convenience we can define this as
\[
\hat{f}(y) = \int_{\mathbb{K}} f(x) \Phi(xy) \, dx
\]
without the conjugation on \( \Phi \). The only difference is that the Fourier inversion formula now reads
\[
f(x) = \int_{\mathbb{K}} \hat{f}(y) \overline{\Phi(xy)} \, dy.
\]
This will be more convenient for our applications.

Let \( f \in S(\mathbb{K}) \) and \( \chi \in \mathcal{X}^{+}(\mathbb{K}^{+}) \). The local theta function is defined to be
\[
\mathcal{Z}(f, \chi) = \int_{\mathbb{K}^{+}} f(x) \chi(x) \, dx.
\]
The joint result we are interested in is the following theorem.
**Theorem 4.23:** Let \( f \in S(\mathbb{R}) \), \( x = \mu_1 \xi \) with \( \mu \) unitary. Then \( Z(f, x) \) is absolutely convergent if \( \mu = \Re(\xi) > 0 \).

**Proof:** The fact that \( \mu \) is unitary gives that we only need to consider

\[
I(f, \sigma) = c \int |f(x)| |x|^{\sigma - 1} \, dx
\]

and show \( I(f, \sigma) < \infty \).

**Case 2:** \( \sigma \) even.

The fact that \( f \) is Schwartz implies \( |f(x)| |x|^{\sigma - 1} \to 0 \)

as \( x \to 0 \) rapidly. Hence \( |x|^{\sigma - 1} \) is integrable around \( 0 \) for any positive \( \sigma \) giving that the integral makes sense around \( 0 \). Thus, the integral is finite as claimed.

**Case 2:** \( \sigma \) real.

In this case we have that \( f \) is locally constant. Thus, given \( x \in \mathbb{R}^n \), s.t.

\[
|f(x)| |x|^{\sigma - 1} = y \quad u \in \mathbb{R}^n.
\]

Since the Haar measure is translation invariant and linear,
it is enough to consider only the f that are the characteristics

functions of the ideal \( \mathcal{m}_k \). Write

\[
\mathcal{m}_k^{\sigma} = 1_{(0,1)^k} = \frac{1}{k!} \mathcal{m}_k^{\sigma - k}.
\]

This gives

\[
\mathcal{Z}(f, x) = c \int \frac{|f(x)|}{|x|^{\sigma - k}} \, dx
\]

\[
= \varepsilon \int \frac{|f(x)|}{|x|^{\sigma - k}} \, dx
\]

\[
= \varepsilon \int \frac{|f(x)|}{|x|^{\sigma - k}} \, dx
\]

\[
= \varepsilon \int \frac{|f(x)|}{|x|^{\sigma - k}} \left( \sum_{k \geq j} \frac{|x_j|^{k-k}}{|1- \omega x_j|} \right),
\]

which is finite for \( \sigma > 0 \).

**Thm 9.34:** Let \( f \in S(k), \quad \mathcal{R} = \mathbb{R}^{1, \infty} \), analytic. If \( \Re(s) > 0 \) satisfies

\[
\varepsilon \in (0, 1),
\]

there is a functional equation

\[
\mathcal{Z}(\hat{f}, x^\sigma) = \mathcal{R}(x, \psi, dx) \mathcal{Z}(f, x^\sigma)
\]

for some \( \mathcal{R}(x, \psi, dx) \) independent of \( f \) which is meromorphic

as a function of \( s \).

Before we prove this theorem, we prove the following result
**Theorem 5.05:** Let \( f, \chi \) be as in Thm 4.24, and let \( S(\chi) \). If \( \sigma \in (0,1) \), we have

\[
Z(f, x) Z(\hat{\sigma}, x') = Z(f, x) Z(\hat{\sigma}, x).
\]

**Proof:** we have

\[
Z(f, x) Z(\hat{\sigma}, x') = \int f(x) \chi(x) \, dx \int \hat{\sigma}(x') \chi(x') \, dx'
\]

\[
= \int f(x) \chi(x) \hat{\sigma}(x') \, dx \int \chi(x') \, dx'
\]

\[
= \int f(x) \hat{\sigma}(x) \chi(x) \, dx \int \hat{\sigma}(x) \, dx.
\]

The measure \( dx \, dx' \) in the product measure is \( R^n \times R^n \), and hence is invariant under the translation \((x, y) \rightarrow (x, x')\). Then, we have

\[
Z(f, x) Z(\hat{\sigma}, x') = \int f(x) \hat{\sigma}(x) \chi(x) \, dx \int \hat{\sigma}(x) \, dx
\]

**Claim:**

\[
\int f(x) \hat{\sigma}(x) \chi(x) \, dx = \int f(x) \hat{\sigma}(x) \chi(x) \, dx
\]

**Proof:** Recall that \( a \, dx = b \, dx \) and

\[
\hat{\sigma}(x) = \int h(x) \psi(x) \, dx.
\]

This gives
\[
\int_{\mathbb{R}^n} f(x_1) \hat{h}(x_2) \text{d}x_1 \text{d}x_2 = c \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} h(z) \psi(x_2 \text{d}z) \text{d}x_2 \right) \text{d}x_1
\]

\[
= c \int_{\mathbb{R}^n} h(x_2) \hat{f}(x_2) \text{d}x_2 \quad \text{(by Fubini's theorem)}
\]

\[
= \int_{\mathbb{R}^n} h(x_1) \hat{f}(x_1) \text{d}x_1 \hat{f}(x_2) \text{d}x_2
\]

as claimed. \hfill \square

This above now completes the proof of the lemma.

Let \( f_0 \in S(\mathbb{R}) \) and set

\[
Y(x) = Y(x, \psi, \text{d}x) \cdot \frac{Z(\hat{f}_0, x)}{Z(f_0, x)}.
\]

The previous lemma shows that this is independent of the "test function" \( \psi \) as we choose. We have

\[
Z(\hat{f}, x) = Y(x, \psi, \text{d}x) \cdot Z(f, x)
\]

do claimed. All that remains to show that \( Y(x, \psi, \text{d}x) \) is in fact a continuous function. We will prove this in the process of proving the following theorem.
Theorem 4.25: Let \( F(x) \) and \( x = f_{1,15} \) in \( \mu \) constant.

Then \( F(x) \) is finite \( \mathcal{E}(x, \Psi(x)) \) that lies in \( \mathcal{E} \) for all \( x \) and satisfies the relation

\[
\mathcal{E}(x, \Psi(x)) = \mathcal{E}(x, \Psi(x)) \frac{L(x)}{L(x)}.
\]

Proof: We split into cases.

Case 1: \( x = \pi \).

Take \( \Psi(x) \) to be Lebesgue measure and set \( \Psi(x) = e^{-i\pi x} \). We know that \( \Psi(x) \) must be of the form \( 1/x \) or \( \sin x \). Suppose that \( \Psi(x) = 1/x \). We take \( \Psi(x) = e^{-i\pi x} \) as our test function.

We have

\[
Z(f(x)) = \int_{\mathbb{R}^x} e^{-i\pi x} 1/x^1 dx x = \int_0^{\infty} e^{-x x^1} x^{x^1} dx
\]

\[
= \pi^{-\frac{x^2}{2}} \int_0^{\infty} e^{-u} u^{x^2-1} du
\]

\[
= \pi^{-\frac{x^2}{2}} \Gamma(s/2)
\]

Thus, we have \( Z(f(x)) = L(x) \) recalling how we defined the Arch. \( L \)-function. We also have (classical formula)

\[
\hat{f}(y) = f(y), \text{ (one uses contour integration to prove this!)}
\]

Hence
\[ Z(f, \chi^y) = \int_{\mathbb{R}^n} f(x) \chi^y(x) \, dx. \]

\[ = L(\chi^y) \] (the is what we first obtain replacing \( x \) by \( \chi^y \)).

Thus, for \( \chi = \chi^1 \), we have

\[ \chi(x) = \frac{L(\chi^y)}{L(\chi)} \]

and so we get \( \chi(x) = \chi(x, \chi^y) \, dx = 1. \)

Assume now that \( \chi = \text{sign } |x|^\frac{3}{2} \). Take \( f(x) = x \, e^{-|x|^2} \);

we have

\[ Z(f, \chi) = \int_{\mathbb{R}^n} x \, e^{-|x|^2} \frac{x}{|x|^{\frac{3}{2}}} \, dx \]

\[ = \int_{\mathbb{R}^n} e^{-|x|^2} \frac{|x|^{\frac{3}{2}}}{|x|^{\frac{3}{2}}} \, dx \]

\[ = \pi \left( \frac{2}{3} \right) \Gamma \left( \frac{2}{3} \right). \]

\[ = L(\chi). \]

Combining integration gives

\[ \hat{f}(y) = iy \, e^{-|y|^2} \]

Thus,

\[ Z(\hat{f}, \chi^y) = i \int_{\mathbb{R}^n} x \, e^{-|x|^2} \frac{x}{|x|^{\frac{3}{2}}} \, dx \]

\[ = i \cdot L(\chi^y). \]
Case 2: $K = C$.

The measure we use is $d\mathcal{L}(x) = \theta dx$. This is under the assumption of Lebesgue measure. Let $\psi(x) = e^{-2\pi i (x \cdot \bar{x})}$. The measure we have chosen is self-dual with respect to this choice, $\bar{\psi} = \psi$. Thus, we take the mean $\mu = 2 \bar{x}$. This is an equivalent mean and is more convenient for our calculations. Any choice of $\nu \in C^*$ can be written as

$$\nu = re^{i\theta} = r e^{i\theta}$$

for some $r \in \mathbb{C}$, $\theta \in \mathbb{R}$. Let

$$f_n(z) = \begin{cases} (2\pi)^{-1} \bar{z} e^{-2\pi i \bar{z} \bar{x}} & \text{for } n > 0 \\ (2\pi)^{-1} \bar{z}^{-n} e^{-2\pi i \bar{z} \bar{x}} & \text{for } n < 0. \end{cases}$$

Exercise: Show

$$f_n(z) = (2\pi)^{-1} \bar{z} e^{-2\pi i \bar{z} \bar{x}}$$

Observe that $d^* \bar{z} = (\frac{1}{2\pi}) d\theta d\bar{z}$ (because of the choice of measure).

For $n > 0$, we have

$$\mathbb{E}(f_n, \chi_{\nu}) = \int_{C^*} f_n(z) \chi_{\nu}(z) d^* \bar{z}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} z^{-n} e^{2\pi i z \bar{x} / \bar{z}} \bar{z} e^{i \theta(\bar{z})} d\bar{z}.$$
\[
\begin{align*}
\frac{1}{r^m} & \int_0^a \int_0^b e^{-\alpha \sqrt{r^2 + s^2}} \, dr \, ds \\
& = \frac{(2\pi)^{\frac{m}{2}}}{\Gamma(s+\delta)} \int_0^\infty e^{-\alpha \sqrt{r^2 + \delta^2}} \, r^{s-\frac{1}{2}} \, dr \\
& = (2\pi)^{\frac{m}{2}} \int_0^\infty e^{-t} t^{s+\delta - 1} \, dt \\
& = (2\pi)^{\frac{m}{2}} \Gamma(s+\delta)
\end{align*}
\]

\[
L(X_{s,n}).
\]

\textbf{Exercise:} Show that for \( n > 0 \) one has

\[
Z(f_n, X_{s,n}) = L(X_{s,n}).
\]

It is clear that \( X_{s,n} \) = \( X_{s,s-n} \) and so one can

run through the same arg. using the above formula for

\( f_n \) to obtain

\[
Z(f_n, X_{s,n}) = \lim_{n \to \infty} L(X_{s,n}).
\]

Thus, we have

\[
Y(X_{s,n}) = \lim_{n \to \infty} \frac{L(X_{s,n})}{L(X_{s,n})},
\]

and so

\[
Z(X_{s,n}) = \lim_{n \to \infty}.
\]
Case 3: $K$ unramified, $\mathrm{char}(K)=0$

Let $K\ell_Q$ be a finite extension. The standard additive character $\psi_p$ is defined on $\mathbb{Q}_p$ as follows. Let $x \in \mathbb{Q}_p$ and let $n$ be the smallest nonzero integer such that $p^n x \in \mathbb{Z}_p$. Let $r \in \mathbb{Z}$ such that $r = p^n x \mod p^n$. Let

$$\psi_p(x) = e^{2\pi i r/p^n}.$$ 

One should check that this is a well-defined character. We get a standard additive character $\psi_K$ on $K$ via

$$\psi_K(x) = \psi_p(x^n).$$

We have that any additive character $\psi$ of $K$ can be written as

$$\psi(x) = \psi_K(x^n)$$

for some $z \in K$.

For any additive character $\psi$ of $K$ and a measure $dx$ that is self-dual with $\psi$, define $m \in \mathbb{Z}$ by

$$m = \inf \left\{ r \in \mathbb{Z} : \psi_p^r(1) = 1 \right\}.$$

This is a finite minimum because $\psi$ is continuous and $\psi(0) = 1$.

We briefly recall a little algebraic number theory.
Recall that the trace map is nondegenerate. Define the dual of $\mathcal{O}$ by

$\mathcal{O}' = \{ x \in K : \text{tr}(x, \mathcal{O}) \in \mathbb{Z}_p \}.$

This is clearly a $\mathbb{Z}_p$-submodule of $K$. Moreover, $\mathcal{O}'$ is a $\mathbb{Z}_p$-module.

Moreover, the different $\mathcal{D}_K$ of $K$ is defined by

$\mathcal{D}_K = (\mathcal{O}')^{-1} = \mathcal{O}^{\mathcal{O}'}.$

Thus, the different is the inverse of the dual of $\mathcal{O}$, not the trace map.

Let $\mu$ be a unitary character of conductor $\infty$,

and set

$\chi_{S, \mu}(x) = \mu \left( \frac{x}{1 \times 1} \right) |x|_S^3.$

Define

$f(x) = \begin{cases} 
\chi_{S, \mu}(x) & \text{if } x \not \in \mathcal{O} \\
0 & \text{if} \quad x \in \mathcal{O}
\end{cases}$

We now split into cases $\alpha = 0$ and $\alpha \neq 0$. 
In this case, we see that \( \mu \) is Thomas\'s measure.

Recall that

\[
\omega^{m}(O - 0) = \frac{1}{\pi^{m/2}} \omega^{n} \delta^{*} x.
\]  \((\star)\)

We then have

\[
Z(f, \chi_{\o{A}}) = \int f(x) \chi_{\o{A}}(x) d^{\pi}x
\]

\[
= \int \Psi(x) \mid x \mid^{\delta} d^{\pi}x \quad (\text{by} \; \Psi \text{-independent of } \omega) \]

\[
= \int \mid x \mid^{\delta} d^{\pi}x \quad (\text{by } (\star))
\]

\[
= \nu \Gamma(\delta, d^{\pi}x) \sum_{k \in \mathbb{Z}^{m}} q^{-k \delta} \quad (\text{by } (\star))
\]

\[
= \nu \Gamma(\delta, d^{\pi}x) \frac{q^{-m \delta}}{1 - q^{-\delta}}
\]

\[
= q^{-m \delta} \nu \Gamma(\delta, d^{\pi}x) L(\pi, A)
\]

Where \( L(\pi, A) = L(\chi_{\o{A}}) \).

This case is \( q_0 \) comes more complicated. We have

\[
Z(f, \chi_{\o{A}}) = \int f(x) \chi_{\o{A}}(x) d^{\pi}x
\]
\[
\mathcal{Z}(f, \chi, n) = \sum_{k \geq m-n} q^{-ks} g(\mu, \Psi_{\text{exp}}(x))
\]

where \( \Psi_{\text{exp}}(x) = \Psi(\chi x) \).

**Lemma 1.24** Let \( \tau \) and \( \sigma \) be unitary chars, \( \tau \) must, or odd with conducting \( \tau' \) and \( \tau'' \) respectively, then we have

1. If \( m < n \), then \( g(\tau, \sigma) > 0 \).

2. If \( m = n \), then

\[
|g(\tau, \sigma)|^2 = c \text{Vol}(\mathcal{O}, d\chi) \text{Vol}(1+\sigma^{-1}, d\mu).
\]

3. If \( m > n \), then

\[
|g(\tau, \sigma)|^2 = c \text{Vol}(\mathcal{O}, d\chi) \left[ \text{Vol}(1+\sigma^{-1}, d\mu) - q^{-1} \text{Vol}(1+\sigma^{-1}, d\mu) \right].
\]
For convenience write \( U = \mathcal{O} \), \( U_0 = 1 + \mathcal{O} \). Assume \( r < n \). Write \( U = \bigsqcup U_a \) as a disjoint union of sets. Then we have

\[
g(z, \sigma) = \sum_{U_a} \int \tau(u) \sigma(u) \, d^* u
\]

and

\[
\sigma(u) = \sigma((1 + \mathcal{O})u) = \sigma(1) \sigma(\mathcal{O}u)
\]

Now observe that for \( u \in U_r \), we can write \( u = 1 + \mathcal{O} \).

Moreover, \( \sigma \) has conductor \( \mathcal{O}^* \mathcal{O} \) and so

\[
\sigma(\mathcal{O}^* \mathcal{O}u) = 1.
\]

Then, we have

\[
g(z, \sigma) = \sum_{U_a} \tau(u) \sigma(u) \int \tau(u) \, d^* u.
\]

We have that \( \tau(u) \neq 1 \) since \( r < n \) and \( \tau \) has conductor \( \mathcal{O}^* \mathcal{O} \). Thus, the integral

\[
\int_{U_r} \tau(u) \, d^* u = \int_{U_r} \tau(u) \mathcal{O} \, d^* u
\]

is 0.

by the orthogonality of characters, and so \( g(z, \sigma) = 0 \).
and so we have \( \sigma \).

Assume now that \( r > n \). Then we have

\[
|g(\tau, \sigma)|^2 = \int \tau(x) \sigma(x) \, dx \int \tau(y) \sigma(y) \, dy \quad (\text{modulo, } \text{rad})
\]

\[
= \int \int \tau(xy) \sigma(y-x) \, dx \, dy.
\]

Also

\[
h(z) = \int \sigma(y(z-1)) \, dy.
\]

Then we have

\[
h(z) = c \int \sigma(y(z-1)) \, dy
\]

Define \( c \cdot dx = 1 \cdot dx \) and \( 1 \cdot dx = 2 \) when \( r > n \).

Thus,

\[
|g(\tau, \sigma)|^2 = \int \tau(z) h(z) \, dx.
\]

Observe that

\[
h(z) = c \int \sigma(y(z-1)) \, dy - c \int \sigma(y(1-z)) \, dy.
\]

\[
= \begin{cases} 
    c \cdot \text{vol}(0, dx) - c q^{-1} \cdot \text{vol}(0, dy) & \text{if } \nu_{y}(z-1) > r \\
    - c q^{-1} \cdot \text{vol}(0, dx) & \text{if } \nu_{y}(z-1) = r \\
    0 & \text{otherwise}
\end{cases}
\]

Thus, we obtain
If \( r = 0 \), then (necessity: \( r > 0 \) since \( r > n \))

\[
|g(r, \sigma)|^2 = \int_{U} z(z) h(z) d^rz
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \int_{U} z(z) d^rz
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \int_{U} d^rz
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \text{vol}(U^*, d^*\sigma).
\]

If \( r > 0 \), then we have

\[
|g(r, \sigma)|^2 = \int_{U} z(z) h(z) d^rz
\]

\[
= \int_{U} z(z) h(z) d^rz
\]

\[
= \int_{U_1} z(z) h(z) d^rz - \int_{U_1 - U_r} z(z) h(z) d^rz
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \text{vol}(U_1, d^*\sigma) + c q^{-1} \text{vol}(U_1, d^*\sigma) \int_{U_1 - U_r} z(z) d^rz
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \text{vol}(U_1, d^*\sigma) - c q^{-1} \text{vol}(U_1, d^*\sigma) \left[ \int_{U_1} z(z) d^rz - \int_{U_r} z(z) d^rz \right]
\]

\[
= c(1 - q^{-n}) \text{vol}(U, d\sigma) \text{vol}(U_1, d^*\sigma) - c q^{-1} \text{vol}(U_1, d^*\sigma) \text{vol}(U_1, d^*\sigma)
\]

\[
= c \text{vol}(U, d\sigma) \left[ \text{vol}(U_1, d^*\sigma) - q^{-n} \int_{U_1} z(z) d^rz \right].
\]

From this point \( \Theta \) and \( \Theta \) follow. \( \square \)
We now use this to compute \( Z(f, x_n) \):

For \( n \gg 0 \), we have

\[
Z(f, x_n) \equiv \sum_{k \leq n} q^{-\frac{k}{2}} g(\mu, \eta_n) \quad (k, \mu \equiv \eta_n \mod 4) \]

\[
= q^{-(m-n)} g(\mu, \eta_m) .
\]

Since \( \Psi \) has conductor \( m \), \( \Psi_{m-n} \) has conductor \( n \). By definition, \( \mu \) has conductor \( n \), and so \( \psi(\mu) \) of the lemma applies to this sharp sum. Thus, \( Z(f, x_n) \) is \( q^{-(m-n)} \) times a nonzero constant and \( n \) has no zeros on pairs. We now compute the other half of the calculation, i.e., the calculation of \( Z(f, x_n^* \).

First we prove the following lemma.

**Lemma 4.2:** Let \( f \) be defined as above. Then \( \hat{f} \) is given by

\[
\hat{f}(y) = \int f(x) \psi(x-y) \, dx = \int \psi(x-y) \, dx
\]

for \( n = 0 \) where \( \psi \) means the characteristic function on the set \( X \).

**Proof:** We have

\[
\hat{f}(y) = \int f(x) \psi(x-y) \, dx = \int \psi(x-y) \, dx
\]
Suppose first that \( n \neq 0 \). If \( y \in \Omega \), then \( \chi(x) \in \Omega'_{n,0} \) and so \( \Psi(x, y) = 1 \). Thus, \( \hat{f}(y) = \text{vol}(\Omega'_{n,0}, dw) \) for \( n \neq 0, y \in \Omega \).

If \( y \in \Omega \), then \( \Psi(x, y) \neq 1 \) for some \( x \). Thus, by the orthogonality of characters, \( \hat{f}(y) = 0 \). Thus, the result is true for \( n \neq 0 \).

Suppose now that \( n = 0 \). If \( y \in \Omega_{0} \), then \( y + iq \in \Omega_{0} \)
and so \( \Psi(x, y) = 1 \). Thus, \( \chi(x) \in \Omega_{0} \). Thus, \( \Psi(x, y) = 1 \) for some \( x \) and so by the orthogonality of characters we get \( \hat{f}(y) = 0 \). Suppose now that \( y \in \Omega_{0} \). Then \( \Psi(x, y) \neq 1 \) for some \( x \) and so \( \hat{f}(y) = \text{vol}(\Omega_{0}, dw) \), as claimed.

We are now finally able to compute \( Z(\mathcal{F}, \chi_{n}) \) and complete the proof of Theorem 4.25. As before, we begin with the case \( n \neq 0 \).

\[
Z(\mathcal{F}, \chi_{n}) = \sum_{x \in K_{0}} \hat{f}(x) \chi_{n,0}(x) d^{v}
\]

\[
= \text{vol}(\Omega_{n,0}, dw) \int_{\Omega_{n,0}} \chi_{n,0}(x) d^{v}
\]

\[
= \text{vol}(\Omega_{n,0}, dw) \int_{\Omega_{n,0}} |x|^{1-\epsilon} d^{v}
\]
\[ Z(f, \chi_{s_0}) = \varphi^{-m_5} \text{vol}(\mathcal{O}^r, dx) \cdot L(\chi_{s_0}) \]

We now combine this with the earlier computation that shows

\[ Z(f, \chi_{s_0}) = \varphi^{-m_5} \text{vol}(\mathcal{O}^r, dx) \cdot L(\chi_{s_0}) \]

To conclude that

\[ Y(\chi_{s_0}) = \frac{Z(f, \chi_{s_0})}{Z(f, \chi_{s_0})} = \frac{\varphi^{-m_5} \text{vol}(\mathcal{O}^r, dx) \cdot L(\chi_{s_0})}{L(\chi_{s_0})} \]

Thus, we see that \( Z(\chi_{s_0}) = \varphi^{-m_5} \text{vol}(\mathcal{O}^r, dx) \), we have the desired equation of Theorem 4.25 for \( n = 0 \).

Assume now that \( n > 0 \). We use the lemma again.

\[ Z(f, \chi_{s_0}^r) = \text{vol}(\mathcal{O}^r, dx) \int_{\mathbb{R}^d} \tilde{\mu}(u) \mu^r(u) \mu^s(u) \]

\[ = \text{vol}(\mathcal{O}^r, dx) \int_{\mathbb{R}^d} \tilde{\mu}(u) \mu^s(u) \]

\[ = \text{vol}(\mathcal{O}^r, dx) \int_{\mathbb{R}^d} \tilde{\mu}^{r-s}(u) \mu^s(u) \]

\[ = \text{vol}(\mathcal{O}^r, dx) \int_{\mathbb{R}^d} \tilde{\mu}^{r}(u) \mu^{s}(u) \]
\[ Z(\Omega, \psi) = \int_{\Omega} \psi(x) \, dx \]

where \( \psi \) is in \( \mathcal{D}(\Omega) \) as well.

Thus, putting everything together we have

\[ E(X_n, \psi, dx) = \frac{\nu(\Omega_n, \psi, dx)}{\nu(\Omega_n, \psi, dx) \cdot \nu(\Omega_n, \psi, dx)} \cdot g(\mu, \psi, dx). \]

This finishes the proof of Theorem 4.25. All that remains to finish the proof of Theorem 4.24.

Observe we have

\[ V(X, \psi, dx) = E(X, \psi, dx) \cdot L(x) / L(x). \]

Thus, it is clear that \( V(X, \psi, dx) \) is a monomorphic function.

We conclude this section with the following observations.

The region of absolute convergence of \( Z(f, \psi) \) in \( \text{Re}(s) > 0 \) and of \( Z(f, \psi^*) \) in \( \text{Re}(s) < 1 \). Thus, the poles of \( Z(f, \psi) \) must be given by the zeros of \( V(X, \psi, dx) \) (in \( \text{Re}(s) = 0 \)).
Moreover, since it is clear that $L(x)$ and $L(x^2)$ have no
zeros, it must be that the zeros of $Y(x, y, (x))$ coincide
with the pole of $L(x)$.

Finally, one should note that for $\nu \neq 0$, the character
$\chi_{\rho_\nu}$ is ramified since it is not trivial on $\mathbb{Q}^\times$. Namely,
$\rho$ is the part that is a character on $\mathbb{Q}^\times$. If $\nu = 0$, then the
character is trivial on $\mathbb{Q}^\times$ and so is unramified. As such,
the calculations of $\nu = 0$ are generally referred to as the "unramified
calculations." As was the case here, the unramified calculations
are generally easier than the ramified ones.
In order to complete a global analysis of zeta integrals, and hence prove the functional equation of Hecke $L$-functions, we will need the Poisson summation formula as well as a generalization known as the Riemann-Roch formula. This can be set up in the geometric setting to give the classical Riemann-Roch formula of elementary algebraic geometry, but we do not pursue that here.

Let $K$ be a number field and set

$$S/\mathbb{A}_K = \bigsqcup_{v} S(K_v)$$

where the restricted tensor product consists of functions of the form

$$f = \otimes_{v} f_v : S(K_v) \times \hat{S}(K_v) \to \mathbb{C},$$

and $\hat{S}(K_v)$ is the dual of the group $S(K_v)$.

We write

$$\tilde{f}(x) = \prod_{v} f_v(x_v)$$

for $x = (x_v) \in \mathbb{A}_K$. Let $\mu$ be a Haar measure on the adeles defining the additive Fourier transform of $S(\mathbb{A}_K)$ by
\[ \hat{f}(y) = \int_{\mathbb{A}_k} f(x) \Psi(xy) \, dx \]

where \( \Psi \) is a non-trivial continuous unitary character on \( \mathbb{A}_k \) at \( \psi_k = 1 \). The measure \( dx \) is normalized to be essentially unitary.

Define

\[ \hat{f}(x) = \sum_{y \in K} f(yx). \]

It is easy to see that for every \( \delta \in K \), we have \( \hat{f}(x+\delta) = \hat{f}(x) \).

We say a function \( f \) is *absolutely and uniformly convergent* if it is absolutely and uniformly convergent on compact sets.

**Lemma 4.27:** For \( f \in S(\mathbb{A}_k) \), we have

\[ \hat{f} |_{\psi_k} = \hat{f} |_{\psi_k}. \]

**Proof:** Let \( \epsilon = K \). Then we have

\[ \hat{f}(\epsilon) = \int_{\mathbb{A}_k} \hat{f}(x) \Psi(tz) \, dx \]

\[ = \int_{\mathbb{A}_k} \left( \sum_{y \in K} \Psi(tz) \right) \, dx \]

By assumption, we have \( \Psi |_{\psi_k} = 1 \), and so \( \Psi(tz) = \Psi(t(1+\epsilon)). \)
Thus, we have

\[ \hat{f}(z) = \sum_{\gamma \in \Gamma \backslash \mathbb{H}} \psi(\gamma z) \Delta(z) \, dz \]

\[ = \int \psi(tz) \, dt \]

\[ = \hat{f}(z). \]

**Lemma 4.28:** Let \( f \in S(\mathbb{H}) \). For every \( \gamma \), we have

\[ \hat{f}(x) = \sum_{\gamma \in \Gamma \backslash \mathbb{H}} \hat{f}(\gamma) \psi(\gamma x). \]

**Proof:** We have just shown that \( \hat{f}_{\Gamma} = \hat{f}_{\Gamma} \). Thus, we have

\[ \sum_{\gamma \in \Gamma \backslash \mathbb{H}} \hat{f}(\gamma) \psi(\gamma x) \]

is uniformly convergent, i.e., we have

\[ \sum_{\gamma \in \Gamma} | \hat{f}(\gamma) | < \infty. \]

Thus, we can apply the Fourier inversion theorem to obtain the result.
Theorem 4.29 (Poisson Summation Formula): Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$
\sum_{y \in \mathbb{R}} \hat{f}(y x) = \sum_{y \in \mathbb{R}} \check{f}(y x)
$$

for all $x \in \mathbb{R}$, i.e., $\check{f} = \hat{f}$.

Proof: Lemma 4.28 gives (plugging in $x=0$):

$$
\hat{f}(0) = \sum_{y \in \mathbb{R}} \hat{f}(y).
$$

However, applying Lemma 4.27 gives $\hat{f}(0) = \hat{f}(y)$

and so

$$
\hat{f}(0) = \sum_{y \in \mathbb{R}} \hat{f}(y).
$$

By definition we have

$$
\check{f}(0) = \sum_{y \in \mathbb{R}} f(y).
$$

Combining these gives the result. \( \blacksquare \)

While this result is very important, we will need a slightly

change form

Theorem 4.30 (Plemann-Pegg Formula): Let $x \in \mathbb{R}_+$ and $f \in \mathcal{S}(\mathbb{R}_+)$. Then

$$
\sum_{y \in \mathbb{R}} f(y x) = \frac{1}{|x|} \sum_{y \in \mathbb{R}} \hat{f}(y x^{-1})).
$$
Proof: Let \( h(y) = f(yx) \) for \( y \in \mathbb{A}_n \) and \( x \) some fixed circle. The fact that \( \phi(x) \) makes it clear that \( \phi(x) \) as well. Theorem 4.29 gives

\[
\sum_{\mathbf{v}} h(x) = \sum_{\mathbf{v}} \hat{h}(x).
\]

However, we have

\[
\hat{h}(x) = \int f(yx) \psi(\xi y) \, dy.
\]

\[
= \frac{1}{|A|} \int f(y) \psi(\gamma_0 x^{-1}) \, dy \quad \text{(change variable: } \gamma \mapsto \frac{1}{x} y\text{)}
\]

\[
= \frac{1}{|A|} \hat{f}(\gamma x^{-1}).
\]

The result is now clear. \( \blacksquare \)
§4.8: Global zeta integrals and functional equation:

Let \( \mathbb{K} \) be a commutative field. Recall that for \( x \in \mathbb{K} \), we defined the standard character \( \psi(x) = e^{2\pi i bx(x)} \). At \( \mathbb{Q} \), we defined

\[
\psi_p(x) = \psi_p(x_{(\mathbb{Q})}) \quad \text{where} \quad \psi_p(x) \quad \text{was defined as follows: For a minimal}\ 
\text{etal} \quad p^n \mathbb{Z} = \mathbb{Z}_p, \quad \text{chose} \quad r \in \mathbb{Z} \quad \text{so that} \quad r \neq p^n \cdot \text{ord}_p(x). \quad \text{Then} \quad \psi_p(x) = e^{2\pi i r/p^n}.
\]

Let \( dx \) be the measure on \( \mathbb{K} \), that is self-dual over \( \mathbb{Q} \). Define the standard character in \( \mathbb{K} \) by

\[
\psi_k(x) = \prod \psi_p(x).
\]

Exercise: Why is \( \psi_k \) well-defined?

As before, we have a map

\[
\begin{array}{ccc}
\mathbb{A}_k & \longrightarrow & \hat{\mathbb{A}}_k \\
Y & \mapsto & \psi_{k,Y}
\end{array}
\]

where as before \( \psi_{k,Y}(x) = \psi_k(yx) \). We have the following result.

Prop. 4.31: The following hold:

1. The group \( \mathbb{A}_k \) is self-dual over the ring \( \mathbb{Z} \) with \( y \mapsto \psi_{k,Y} \).
3. The character $\psi_n$ is trivial on $K$ and so can be viewed as a character on $M_k/K$.

3. The Pontryagin dual of $M_k/K$ (resp. $K$) can be identified with $K$ (resp. $M_k/K$). Explicitly, this can be realized by the map

$$x \in K \rightarrow \Psi(x, (M_k/K)^\vee).$$

Hence, the self-duality $\Psi$ gives that the translation $\Psi_{x,y}$ is trivial if $x, y \in K$.

4. If $\psi$ is any character on $M_k/K$, then $\psi$ has conductor $\Theta_{\psi}$ for $\psi \in \psi$.

**Proof:** Exercise.

Let $dx = \prod dy_v$. This measure is self-dual and

$$\Psi_K$$

and satisfies

$$d(\alpha x) = |\alpha| d\alpha$$

for all $\alpha \in M_k$.

Let $\chi$ be an idèle class character. Given $f \in SC^0(K)$,

the global theta function is defined by
\[ Z(f, x) = \int_{x}^{\infty} f(v) d^{*}x. \]

We normalize the measure \( d^{*}x \) as follows. For \( u \) and \( v \) such that \( d^{*}x = \frac{dv}{v^{1/2}} \), for \( u \) and \( v \) such that we take \( c_x = \frac{q_{v}}{q_{v-1}} \) and so \( d^{*}x = \frac{q_{v}}{q_{v-1}} \frac{dv}{v^{1/2}} \). Let \( x' = x^{-1/2} \) as in the third case.

**Exercise:** Show that \( Z(f, x) \) is uniformly convergent for \( s - \Re(s) > 1 \), when \( x = e^{-\mu t} - t \) with \( \mu \) unitary. The function \( Z(f, x) \) is holomorphic in the domain \( \Re(s) > 1 \).

**Theorem 4.32:** The function \( Z(f, x) \) extends to a meromorphic function of \( s \) and satisfies the functional equation

\[ Z(f, x) = Z(f, x^{-1}). \]

The extended function \( Z(f, x) \) is in fact holomorphic everywhere except when \( \mu = 1/2 \) with \( \tau \in \mathbb{R} \). In this case, it has simple poles at \( s = 1/2 \) and \( s = 1/2 \) with residues given by

\[ -\text{Vol} (\mathcal{J}_{x}(K_{r})f_{K}) \text{ and } \frac{1}{2} \text{Vol} (\mathcal{J}_{x}(K_{r})f_{K}) f(0). \]
When we recall \( J_h : \text{ker} \, \text{ker} : M^r \rightarrow M^r \), i.e., \( J_h \) is the compact part of \( M^r \).

Before we prove this theorem, we need the following lemma.

**Lemma 4.33:** Define

\[ Z_t(f, x) = \int_{J_k} f(tx) \, X(tx) \, d^r x \]

where \( t \in \mathbb{R}_+ \) and we take the product \( tx \) to mean \( tx \) in one of the relevant components and \( x \) everywhere else. Then \( Z_t(f, x) \) satisfies the functional equation

\[ Z_t(f, x) = Z_{t^2}(f, x^2) + \hat{f}(0) \int_{J_k} X^r(2tx) d^r x - f(0) \int_{J_k} X(tx) d^r x. \]

**Proof:** We have

\[ Z_t(f, x) = \int_{J_k} \left( \sum_{\alpha \in \mathbb{K}} f(a \cdot x) \right) X(tx) \, d^r x \quad \text{(Recall \( X \) is trivial)} \]

\[ = \int_{J_k} X(tx) \, d^r x \sum_{\alpha \in \mathbb{K}} f(a \cdot x). \]

We would like to apply Riemann - Ricci, but our sum is over...
\[ Z_k(f, x) + f(0) \int_{J^k/k^k} x(lx) d^*x, \]

Which is,

\[ \int_{J^k/k^k} x(lx) d^*x \sum_{a \in k} \hat{f}(at^*x^{-1}). \]

We now apply Riemann - Lind to obtain that this is equal to

\[ \int_{J^k/k^k} x(lx) d^*x \int_{J^k/k^k} \frac{1}{|x|} \sum_{a \in k} \hat{f}(at^*x^{-1}). \]

Thus,

\[ Z_k(f, x) + f(0) \int_{J^k/k^k} x(lx) d^*x = \int_{J^k/k^k} x(lx) d^*x \sum_{a \in k} \hat{f}(at^*x^{-1}) \]

\[ = \int_{J^k/k^k} x(lx) d^*x \sum_{a \in k} \hat{f}(at^*x^{-1}). \]

However, this equals

\[ Z_{k^k}(\hat{f}, x^k) + f(0) \int_{J^k/k^k} x^k(lx^k) d^*x. \]

Thus, the claimed formula holds.
Prm. (Thm 4.3.2): Observe that we have

$$Z(f,x) = \int_0^\infty Z_t(f,x) \frac{1}{t} \, dt.$$ 

Write

$$Z(f,x) = \int_0^1 Z_t(f,x) \frac{1}{t} \, dt + \int_1^\infty Z_t(f,x) \frac{1}{t} \, dt.$$ 

(This should be looking familiar to the Dirichlet L-function cost now!)

The integral \( \int_1^\infty Z_t(f,x) \frac{1}{t} \, dt \) is equal to

$$\int f(x \frac{y}{x^s}) \, dx.$$ 

\(x \frac{y}{x^s} \in (\mathbb{N}, x)\)

This integral converges normally for all \( s \) (Recall \( f \) is a Schwartz–Bruhat function.) We also have

$$\int_0^1 Z_t(f,x) \frac{1}{t} \, dt = \int_0^1 Z_{t^{-1}}(f,x) \frac{1}{t} \, dt + E(f,x)$$

where \( E \) is given by the previous lemma as

$$E = \int \left[ \frac{\hat{f}(0)}{x^s} \right] \int \chi(x) \, dx - \left\{ \int \chi(x) \, dx \right\}^2 \int \frac{x^s}{x^s} \, dx.$$ 

Making the substitution \( t \to t^{-1} \) we have

$$\int_0^1 Z_{t^{-1}}(f,x) \frac{1}{t} \, dt = \int_0^\infty Z_t(f,x) \frac{1}{t} \, dt.$$
which is convergent for all $s$ as was observed above.

The orthogonality of characters gives that

$$
\int_{J_{x/k}} X(x) d\tau \quad \text{and} \quad \int_{J_{x/k}} Y(x) d\tau,
$$

are zero if $X$ is continuous on $J_{x/k}$ and so $E = 0$ in this case as well. Now we deal with the case that $X$ is trivial on $J_{x/k}$. In this case we must have $X = 1$ if $x = i \pi$ when $s = s_i$ for some $i \in \mathbb{R}$. In this case we have

$$
E = \int_0^1 \left[ \hat{f}(0) e^{s_i \tau} \text{vol}(J_{x/k}) - f(0) e^{s_i \tau} \text{vol}(J_{x/k}) \right] \frac{1}{\tau} d\tau
$$

$$
= \text{vol}(J_{x/k}) \left[ \frac{\hat{f}(0)}{s_i - 1} - \frac{f(0)}{s_i} \right].
$$

Then $E$ is a natural function and so we obtain that $Z(f, \chi)$ has meromorphic continuation to all of $C$. In fact, it is holomorphic everywhere if $\mu \neq 1, i \pi$ and when $\mu = 1, i \pi$, the pole can at $s = i \pi$ and $s = 1 + i \pi$

will remain.

$$
-\text{vol}(J_{x/k}) \hat{f}(0) \quad \text{and} \quad \text{vol}(J_{x/k}) \hat{f}(0)
$$

respectively.
We have

$$Z(f, x) = \int_{-\infty}^{\infty} Z_\varepsilon(f, x) \frac{1}{\varepsilon} \, dt + \int_{-\infty}^{\infty} Z_\varepsilon(\hat{f}, x) \frac{1}{\varepsilon} \, dt + E(f, x)$$

$$= \int_{J_k} \int \hat{f}(tx) \chi(tx) \, dx \, dt + \int_{J_k} \int \hat{f}(tx) \chi(tx) \, dx \, dt + E(f, x)$$

Moreover, we have

$$\hat{f}(x) = \hat{f}(-x)$$

and

$$(\chi(tx))' = x$$

(cheer these!)

and so

$$Z(\hat{f}, x') = \int_{-\infty}^{\infty} Z_\varepsilon(\hat{f}, x') \frac{1}{\varepsilon} \, dt + \int_{-\infty}^{\infty} Z_\varepsilon(\hat{f}, (x') \chi) \frac{1}{\varepsilon} \, dt + E(\hat{f}, x')$$

$$= \int_{J_k} \int \hat{f}(tx) \chi(tx) \, dx \, dt + \int_{J_k} \int \hat{f}(-tx) \chi(tx) \, dx \, dt + E(\hat{f}, x')$$

However, our formula for $E$ shows that $E(f, x) = E(\hat{f}, x')$.

The fact that $\chi$ is trivial on $e^x$ implies that $\chi(tx) = \chi(-tx)$

and so we obtain

$$Z(f, x) = Z(\hat{f}, x')$$

as desired.
We now recall from 4.21, the motivation for studying these

guts integrals.

**Theorem 4.21:** Let \( \chi \) be a unitary ideal class character. The function

\[ \Lambda(s, \chi) \]

admits a meromorphic continuation to \( \mathbb{C} \) and
satisfies the functional equation

\[ \Lambda(1-s, \chi^v) = \varepsilon(s, \chi) \Lambda(s, \chi) \]

where

\[ \varepsilon(s, \chi) = \prod_v \varepsilon_v(s, \chi_v, 1/\chi) \]

is defined by Theorem 4.25. Moreover, the meromorphic
continuation is entire unless \( s = 1- \frac{r}{2} \) for \( r \in \mathbb{Z} \), in
which case \( \varepsilon \) poles at \( s = \frac{r}{2} \) and \( s = 1+ \frac{r}{2} \) with
residues \(- \text{vol}(\Omega_{K_v})\) and \( \text{vol}(\Omega_{K_v}) \cdot \text{Nm}(\mathfrak{a}_v)^{1/2} \).

**Proof:** We omit calculation of the actual residues as it amounts
to a calculation we do not wish to include. The interested
reader is encouraged to supply these missing calculations.

Choose \( f = \mathfrak{a} \in S(M_K) \). Then we have

\[ Z(f, x^v) = \prod_v Z(f, x_v^v). \]

We can now apply our global and local functional
\[ 1 = \prod \frac{\delta(x_y) L_1(1, s, x_y)}{L_y(s, x_y)} = \frac{\delta(s, x) A(s, x)}{A(s, x)} \]

Thus, if we can show that \( A(s, x) \) is indeed meromorphic we have our functional equation and meromorphic continuation.

We will have that \( A(s, x) \) is meromorphic if we can show there exists a function \( f = \theta_s + s! A_x \) so that

\[ Z(f, x_1, x_2) = h(s, x) A(s, x) \]

for a meromorphic function \( h \) whose we already know \( Z(f, x_1, x_2) \) is meromorphic. However, locally we have already seen that

\[ Z(f, x_1, x_2) = h'(s, x) L_y(s, x) \]

where \( h \) is entire and everywhere nonzero. In fact,

\[ h = 1 \] for almost all \( x \). If so, we saw \( h = 1 \).

In the previous case, we saw

\[ Z(f, x_3, x) = q^{-s} \nu_1(\varphi^2, \sigma^2) L_y(x_3, x) \]

and
\[ Z(f_v, \chi_v^\infty) = 2^{-(m-1)/2} g(-\chi_v, \psi_{\text{red}}(\infty)) L_v(1, \chi_v). \]

Since \( \chi_v \) is of c.r.e., we have that \( f_v \) is a class function in \( \chi_v \), and thus \( f = \oplus f_v \in S(M_k) \) is
defined. This gives the meromorphic continuation of \( L(s, \chi) \) and using our previous result on \( Z(f, \chi) \).

Along with the fact that \( h \) is entire, one can calculate the poles and residue of \( \Lambda(s, \chi) \).

Finally, we conclude by using class field theory to
relate the \( L \)-functions of idele class characters to those
of Artin \( L \)-functions of \( 1 \)-dimensional representations
of abelian Galois groups. We have the following

Theorem.

\textbf{Theorem 4.34:} Let \( K/k \) be an abelian extension of number fields
with conductor \( f := f(K/k) \). Let \( \rho \) be an irreducible
character of \( \text{Gal}(K/k) \) (i.e., a \( 1 \)-dim rep. of \( \text{Gal}(K/k) \)).

Via GCF++ there exist an idele class character \( \chi \),
so that the $L$-functions satisfy

$$L(v, p, \pi_{1, \omega}) = \left( 1 - \frac{1}{\omega^s} \right) L(\pi, X_p)$$

where $S = \{ v : \pi(v) = 1 \}$.

**Proof:** Recall that the representation $\pi$ is given by a vector space $V$ with

$$p: \text{Gal}(k_{1, \omega}) \to \text{GL}(v).$$

In our case, we have $v = \mathbb{C}$ and so

$$p(\sigma) v = p(\sigma) v,$$

i.e., it is just multiplication by $p(\sigma)$ since $\text{GL}(v) = \mathbb{C}^\times$.

We know that

$$V \neq \{0\} \iff V \text{ is ramified} \iff I(v) \neq \{1\}.$$ 

If $p(I(v)) \neq \{1\}$, then $V^I(v) = \{0\}$ and so

$$L_v(s, p, \pi_{1, \omega}) = 1 \quad \text{in this case.}$$

We have $p(I(v)) = \{1\}$, then $V^I(v) = \mathbb{C}$ and we obtain

$$L_v(s, p, \pi_{1, \omega}) = \det \left( 1 - p(F_{\text{red}}) N \omega^{-s} \right) = 1 - p(F_{\text{red}}) N \omega^{-s}.$$
Thus, we have

\[ L(s, p, \pi_n) = \prod_v L_v(s, p, \pi_n) \]

or

\[ (1 - \rho(\text{Frob}_v) N v^{-s})^{-1} \]

On the other hand,

\[ L(s, \pi_p) = \prod_v L_v(s, \pi_p) \]

\[ = \prod_{v \nmid \infty} L_v(s, \pi_p) \]

\[ = \prod_{v \nmid \infty} (1 - \chi_p(v) N v^{-s})^{-1} \]

discriminant by

when we have used that \( \chi_p \) has conductor \( \infty \) by Prop 4.19.

For \( v \nmid \infty \), we have via GCF that \( v \rightarrow \text{Frob}_v \) under

the global action map, and so \( \chi_p(v) = \rho(\text{Frob}_v) \). Thus

we have the result. \( \blacksquare \)

Now let \( \rho: \text{Gal}(E/k) \rightarrow \mathbb{C}^\times \) be any 1-dim rep. Let

\( E(\rho) = \text{ker}(\rho) \), i.e., the fixed field of the kernel of \( \rho \). Then

by Galois theory we have

\[ \rho: \text{Gal}(E(\rho)/k) \rightarrow \mathbb{C}^\times. \]
In this case, we must have \( p(2v) = 1 \) for all \( v \neq f \)
and so \( S = \emptyset \) in the previous theorem. Thus, we have

\[
L(s, \mathfrak{p}, E(p)_{\mathfrak{m}}) = L(s, \mathfrak{p})
\]

and in this case \( \mathfrak{p} \) is primitive with conductor \( \mathfrak{m} \). It is also clear from the proof of Theorem 4.34 that

\[
L(s, \mathfrak{p}, E(1)_{\mathfrak{m}}) = L(s, \mathfrak{p}, E(1)_{\mathfrak{m}}).
\]

Thus, the holomorphic continuation of \( L(s, \mathfrak{p}) \) gives

that of \( L(s, \mathfrak{p}, E(1)_{\mathfrak{m}}) \) as desired. (Note \( \mathfrak{p} \) is not in

the form \( 1 + \mathfrak{m} \) since it is necessarily of finite

order.) By defining the factor \( L(s, \mathfrak{p}, E(1)_{\mathfrak{m}}) \) for \( v \vDash \mathfrak{m} \)

so that they match up to those in \( L(s, \mathfrak{p}, \mathfrak{m}) \) as they

should, one also obtains the functional equation for

Artin \( L \)-functions of \( 1 \)-dim. reps. from the

functional equation of \( L(s, \mathfrak{p}) \).