Introduction to modular symbols

§1 Modules forms:

Let \( \mathcal{H}^* = \mathcal{H} \cup \mathcal{P}(\mathbb{Q}) \).

Let \( \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \subseteq SL_2(\mathbb{Z}). \)

\[ \Gamma_0(N) \mathcal{H}^* \mathcal{H}^* \text{ via } (a \ b \ c \ d) \cdot z = \frac{az + b}{cz + d}. \]

Let \( f : \mathcal{H}^* \to \mathbb{C} \) be a holomorphic function s.t.

\[ f(gz) = (cz + d)^k f(z), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

Then \( f \) is a modular form of weight \( k \) and level \( N \).

\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) \]

\[ f(Tz) = f(z) \Rightarrow f \text{ has a Fourier expansion of the form} \]

\[ f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}. \]

If \( a_0 = 0 \) we say \( f \) is a cusp form.

\[ M_k(\Gamma_0(N)) = M_k(N) = \text{complex vector space of modular forms of weight } k \text{ and level } \Gamma_0(N). \]

The subspace of cusp forms is denoted \( S_k(\Gamma_0(N)). \)

\[ M_k(N) \text{ comes equipped with linear operators} \]

\[ T_n : M_k(N) \to M_k(N), \quad n \in \mathbb{N}, \]

\[ S_k(N) \to S_k(N) \]

We can find a basis for \( M_k(N) \) consisting of
simultaneous eigenforms eigenvalues for \( T_n \), \((n,N)=1\).

**Goal:** Compute element of \( \Gamma_0(N) \).

### 5.2 Modularity Symbols:

Let \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \).

Then

\[
X(\Gamma) := \Gamma \backslash \mathcal{H}^*
\]

is called a modular curve.

If \( \Gamma = \Gamma_0(N) \), then \( X_0(N) := X(\Gamma_0(N)) \).

**Example:**

![Diagram of \( X_0(1) \)](image)

Let \( M_2(\Gamma) = \text{Hom}_\mathbb{C}(\mathbb{C}, \mathbb{C}) \).

Let \( \alpha, \beta \in \mathcal{H}^* \). Let \( \gamma, \beta \) be a path from \( \alpha \) to \( \beta \)

\( \gamma \in \mathcal{H}^* \).

Let \( \delta_1, \beta \) be the image of \( \gamma \) in \( X(\Gamma) \).
\[ \{ \alpha, \beta \} \Gamma \] determines an element of \( \text{Me}(\Gamma^*) \) via

\[ f \mapsto \int_\alpha^\beta 2\pi i \text{tr}(f) \text{det} \, \mathrm{d}z. \]

We denote this map by \( \{ \alpha, \beta \} \Gamma \) as well; it should be clear from context. These are modular symbols.

**Properties of modular symbols:**

1. \( 3\alpha, \alpha_1 \Gamma = 0 \)
2. \( 3\alpha, \beta_1 \Gamma + 3\beta, \alpha_1 \Gamma = 0 \)
3. \( 3\alpha, 3\beta_1 \Gamma + 3\beta, \alpha_1 \Gamma = 0 \)
4. \( 3\beta, \alpha_1 \Gamma = 3\beta, \beta_1 \Gamma \) \( \forall \gamma \in \Gamma \)
5. \( 3\alpha, \beta_1 \Gamma = 3\beta, \alpha_1 \Gamma \) \( \forall \gamma \in \Gamma \)
6. \( 3\alpha, \beta_1 \Gamma = 3\alpha, \beta_1 \Gamma \) \( \forall \gamma, \gamma_2 \in \Gamma \)

These properties define the modular symbol up to \( \Gamma \)-equivariance. We use \( 3\beta, \alpha_1 \Gamma = 3\beta, \beta_1 \Gamma \), which is invariant under \( \Gamma \).

**Triangulating \( \mathcal{H} \) (simplicial complex):**

- **Vertices:** \( \mathbb{P}^1(\mathbb{C}) \)
- **Edges:** \( \{ \frac{a}{b}, \frac{c}{d} \} = \{ g(a, c), g(c, c) \} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \)
- **Triangles:** \( \text{SL}_2(\mathbb{Z}) \) orbits of \( \{ \frac{a}{b}, \frac{c}{d} \} \) given by \( \{ 0, \infty \}, \{ \infty, \infty \}, \{ 0, 0 \} \) and \( \{ \text{ST}^2(0), \text{ST}^2(\infty) \} \)

Replace \( 3\alpha, \beta_1 \Gamma \) by \( 3\gamma, \beta_1 \Gamma \) to obtain triangulation on \( X_\Gamma \).

**Notation:** \( (g) := 3g(a, c) \Gamma, \beta_1 \Gamma \quad \forall g \in \text{SL}_2(\mathbb{Z}) \)

**Relations:** 1. \( (g) = (g', g) \quad g' \in \Gamma \)
2) $(9) + (95) = 0$

3) $(9) + (g5 + (g5 + 1) + (g5 + 2) = 0$.

Define: $C(\Gamma) := \mathbb{Z}[\Gamma / \text{SL}_2(\mathbb{Z})]$,

$B(\Gamma) := \langle \text{relation } (2), \text{ relation } (3) \rangle \mathbb{Z}$,

$Z(\Gamma) := \ker(\mathcal{D}: C(\Gamma) \to \mathbb{Z}[\Gamma / \text{P}^1(\mathbb{Z})])$

$\langle 9 \rangle \to \Gamma g(\infty) \Gamma_p - \langle g(9) \rangle _p$

$$\begin{align*}
\text{rank} 2g_0 + 3 & \quad \begin{array}{c}
\mathbb{Z}\text{-module} \quad \{ C(\Gamma)/B(\Gamma) \} \\
\mathcal{M}_2(\Gamma)^\nu
\end{array} \\
g_0 = \text{genus}, & \quad g_0 = \# \text{cusps}.
\end{align*}$$

$$\begin{align*}
\text{rank} 2g_0 & \quad \begin{array}{c}
\mathbb{Z}\text{-module} \quad \{ Z(\Gamma)/B(\Gamma) \} \\
S_2(\Gamma)^\nu
\end{array}
\end{align*}$$

Act $\Gamma = \Gamma_0(N)$.

Define: A Manin - symbol is an element of $\text{P}^1(\mathbb{Z}/N\mathbb{Z})$,

where $\text{P}^1(\mathbb{Z}/N\mathbb{Z}) := \{ (x, y) + (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(x, y, N) = 1 \}/(\mathbb{Z}/N\mathbb{Z})^\times$.

Prop: There is a bijection $\Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z}) \to \text{P}^1(\mathbb{Z}/N\mathbb{Z})$

$$\begin{align*}
\begin{array}{c}
\left( \begin{array}{cc}
ab & c \\
d & d
\end{array} \right) \mapsto (c; d).
\end{array}
\end{align*}$$

We may also view $(c; d)$ as a modular symbol via

$$\begin{align*}
\begin{array}{c}
( c; d) \mapsto (a b) = 9 \mapsto (9). \\
\end{array}
\end{align*}$$

We have an action

$$\text{SL}_2(\mathbb{Z}) \times \text{P}^1(\mathbb{Z}/N\mathbb{Z})$$

$$(x; y) \left( \begin{array}{cc}
ab & c \\
d & d
\end{array} \right) = (ax + cy : bx + dy)$$.
In particular,
\[(x:y)_S = (y:-x)\]
\[(x:y)_S T = (y:y-x)\).

Our relations become:
2) \((c:d) + (-d:c) = 0\)
3) \((c:d) + (d:d-c) + (d-c;-c) = 0\).

Our \(S\) map becomes
\[
S : (c:d) \mapsto \left[\begin{array}{c} \alpha_c \\ \beta_d \end{array}\right]_r - \left[\begin{array}{c} \alpha_d \\ \beta_c \end{array}\right]_r.
\]

C(N) := \(\mathbb{Z}[\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})]\)

B(N) := \langle \text{relation (2)}, \text{relation (3)} \rangle_{\mathbb{Z}}

Z(N) := \ker S.

\[
C(N)/B(N) \leftrightarrow M_2(N)^V
\]

\[
(C(N)/B(N))^V := \left\{ \lambda : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z} \mid \begin{array}{l}
\lambda((c:d)) + \lambda((-d:c)) = 0 \\
\lambda((c:d)) + \lambda((d:d-c)) + \lambda((d-c;-c)) = 0
\end{array} \right\}
\]

§3 Graph theoretic view:

Vertices: \(\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})\)

place a blue edge \(p = 95\)

red edge if \(p = 95ST\) or \(p = 9(ST)^2\)

Example: \(N = 2\)
$\infty = \left\lceil 0, 1 \right\rceil$

$\mathbb{Z}/\mathbb{Z}$

Label our vertices from $\mathbb{Z}$.

1) Labels of two vertices connected by a blue edge sum to $0$.

2) Labels of three vertices by connected by a red edge sum to $0$.

Let $L(N) = \exists \lambda : \mathbb{P}^1(\mathbb{Z}/\mathbb{Z}) \to \mathbb{Z} : \lambda(p) + \lambda(ps) = 0$

$\lambda(p) + \lambda(psT) + \lambda(p(st)^2) = 0$.

Observe

$L(N) = \left( C(N)/B(N) \right)^V$.

$\lambda$ red

... blue

Let the operations act on $L(N)$ via

$(T_{k, \lambda})(x:y) = \sum_{\text{gcd}(ax + cy, bx + dy, N) = 1} \lambda \left( \frac{a}{c}, \frac{b}{d} \right)$.

$\text{gcd} = 1$

$ad - bc = 1$
Example: \( N = 11 \)

\[
\begin{array}{c}
(0) \quad (0) \quad (0) \quad (0) \quad (0) \\
\infty \quad (0) \quad (0) \quad 6 \quad \ldots \quad 9 \\
(1) \quad \cdots \quad 10 \\
0 \quad (0) \quad (0) \quad (0) \\
\end{array}
\]

\[
\begin{array}{c}
(6) \\
(0) \\
(2) \\
(1) \\
(3) \\
(4) \\
(5)
\end{array}
\]

\[
\dim \mathbb{S}_2 \left( \Gamma_0(11) \right) = 2
\]

\[
\left( T_2 \lambda \right)(4) = \lambda \left( \left[ \begin{array}{c}
4 \\
1
\end{array} \right] \right) + \lambda \left( \left[ \begin{array}{c}
4 \\
0
\end{array} \right] \right) + \lambda \left( \left[ \begin{array}{c}
2 \\
1
\end{array} \right] \right) + \lambda \left( \left[ \begin{array}{c}
4 \\
0
\end{array} \right] \right)
\]

\[
= \lambda(2) + \lambda(8) + \lambda(6) + \lambda(8)
\]

\[
= 0 + \lambda(a) - \alpha = -2 \alpha
\]

Thus, \( \alpha \) is an eigenvalue of 2.

This graph corresponds to

\[
f(\tau) = q - 2q^2 - q^3 + O(q^4) \in \mathbb{S}_2(11).
\]