Conic Theta Functions:

joint work w/ A. Folsom and S. Robins (2011/12)

Plan:
1. Solid angles and conic theta functions
2. Non-modularity
3. Modularity

Recall:

Def: Let \( w_1, \ldots, w_d \in \mathbb{R}^d \) be a basis. Then
\[
K = \sum_{j=1}^{d} \mathbb{R}^+ w_j.
\]
This is called a polyhedral cone with edges \( w_1, \ldots, w_d \).

Ex: \( P_3 \) or that \( K_0 = \mathbb{R}^d_{\geq 0} \).

Def: One defines the solid angle \( wik \) at the center of \( K \)
(= origin) by
\[
wik := \frac{\text{vol}(K \cap S^{d-1})}{\text{vol}(S^{d-1})}
\]
Remark: i) One has \( 0 < W_k < 2 \).

ii) If \( d = 2 \), then \( W_k = \Theta/2\pi \) where \( \Theta \) is the usual 2-dim. angle measured in radians.

Lemma 1: \( W_k = \int e^{-\pi \| x \|^2} \, dx \)

Proof: Use polar coordinates, i.e., \( x = rs \) with \( r = \| x \| \)

and \( s \in S^{d-1} \). Then \( dx = r^{d-1} \, dr \, ds \). Therefore,

\[
\int_{\mathbb{R}^d} e^{-\pi \| x \|^2} \, dx = \int_0^\infty e^{-\pi r^2} \, dr \int_{S^{d-1}} ds \\
\int_{\mathbb{R}^d} e^{-\pi \| x \|^2} \, dx = \int_0^\infty r^{d-1} e^{-\pi r^2} \, dr \int_{S^{d-1}} ds \\
= \frac{\text{vol}(S^{d-1})}{\text{vol}(S^{d-1})} = W_k. \quad \Box
\]

Def: Let \( L \subseteq \mathbb{R}^d \) be a (full) lattice, \( K \) polyhedral cone.

Then

\[
\Phi_{K,L}(z) = \sum_{m \in L} e^{\pi i m^* z} \quad (z \in \mathbb{C})
\]

is called a conic theta function.

Lemma 2: Write \( L = A Z^d \) where \( A \in G L_d(\mathbb{R}) \). Then

\[
\lim_{t \to 0} t^{d/2} \Phi_{K,L}(ct) = \frac{W_k}{\det A}.
\]
**Proof:** Use Riemann sum: \( f(x) = e^{-\pi i x \cdot \mathbf{h}} \quad (x \in \mathbb{R}^d) \)

\[
W_k = \frac{1}{k} \sum_{m \in \mathbb{Z}^d} \frac{1}{(2\pi)^d} \int f(x) \, dx,
\]

where \((2\pi)^d\) is the \(d\)-dimensional volume of a small cube of side length \(2\pi\), intersecting in sets of measure zero only and comprising \(k\). Choose \(h = \frac{1}{k}\).

Then

\[
W_k = \lim_{h \to 0} \frac{1}{k} \sum_{m \in \mathbb{Z}^d} f(x(m))
\]

\[
= \lim_{h \to 0} \frac{1}{k} \sum_{m \in \mathbb{Z}^d} \frac{1}{(2\pi)^d} e^{-\pi i x(m) \cdot \mathbf{h}}.
\]

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2) **Non-modularity**

Recall: \(L \subseteq \mathbb{R}^d\) lattice, write \(L = A \mathbb{Z}^d\) with

\(A \in \text{GL}(\mathbb{R})\), \(B = A^T A\). We have a quadratic form

\[
Q(x) = x^T B x \quad (x \in \mathbb{R}^d)
\]

the associated positive definite quadratic form, assume \(B\) is even integral. Then

\[
V_L(z) = \sum_{m \in L} e^{\pi i m \cdot z}
\]

\[
= \sum_{m \in \mathbb{Z}^d} e^{\pi i Q(m) z}
\]

\[
= 1 + \sum_{k=1}^{\infty} \left(\frac{c(k)}{k} q^k \right) \quad (q = e^{2\pi i z})
\]
(where \( c(I) = \# \{ n \in \mathbb{Z}^d : \frac{1}{2} a(I) - n \} \).) 

is a modular form of weight \( \frac{d}{2} \) on \( \Gamma_0(N) \) where \( N = \text{level of } \mathcal{Q} \) (smallest pos. integer \( M \) s.t. \( M \mathcal{B}^{-1} \) is even integral), and \( \Gamma_0(N) = \left\{ (a b) \in SL_2(\mathbb{Z}) : N | c \right\} \). 

This essentially means that 

\[
\mathcal{U} \left( \frac{a \mathcal{E} + b}{c \mathcal{E} + d} \right) = \left( \begin{array}{cc} d & b \\ c & d \end{array} \right) \frac{d}{2} \mathcal{U} \left( \frac{a b}{c d} \right) \mathcal{U} \left( \frac{a b}{c d} \right) \quad \forall (a b) \in \Gamma_0(N).
\]

In general, one would not expect that \( \mathcal{F}_{k, \nu}(z) \) are modular!

**Question:** How to prove this? How "non-modular" are they?

**Theorem 1:** 1) Suppose that \( \mathcal{F}_{k, \nu} \) is a modular form of integer or half-integer weight \( k \) on \( \Gamma_0(M) \) for some \( M \in \mathbb{N} \). Then necessarily one must have \( k = \frac{d}{2} \).

2) Suppose that \( \frac{k}{2} \notin \mathbb{Q} \). Then \( \mathcal{F}_{k, \nu} \) is **NOT** a modular form of weight \( \frac{d}{2} \).

**Proof:** 1) Suppose \( \mathcal{F}_{k, \nu} \) is modular of weight \( k \). Then it is

\[
\lim_{t \to 0} t^k \mathcal{F}_{k, \nu}(ct) = b.
\]

exists and is finite. By Lemma 2,
\[
\lim_{t \to 0} t^{d/2} \mathbb{E}_{k,t} (ct) = \frac{\omega_k}{|\det A|} > 0.
\]

1. if \( k < d/2 \) \( \Rightarrow b_0 = 0 \) #

2. if \( k > d/2 \) \( \Rightarrow b_0 = 0 \). Let \( a(n) = n^k \) Fourier coefficient of \( \overline{h}_k \) \( (n > 0) \). As in particular \( a(0) = 1 \).

Let \( D(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \) \( (Re(s) > 1) \). the

Merk L-series attached to \( \overline{h}_k \). By here we know \( D(s) \) has meromorphic continuation to \( \mathbb{C} \)

and is holomorphic except for a possible simple pole at \( s = b_0 = 0 \). Thus, \( D(s) \) is hol. on \( \mathbb{C} \). By

Laplace's theorem, since \( a(n) \to 0 \) \( \forall n \to \mathbb{C} \), \( D(s) \)

must converge \( \forall s \in \mathbb{C} \), so

\[ a(n) = O(n^{\varepsilon}) \quad \forall \epsilon \in \mathbb{R}. \]

By Schmidt's \( \varepsilon \), (2012) \( \Rightarrow b(n) / (n \varepsilon) \) are the

coeff. of a modular form of \( \text{at} \ k \) and

\[ b(n) = O_f \left( \frac{1}{(n^{1/2} + 2)} \right) \quad (\varepsilon > 0) \]

\( \forall \varepsilon > 0 \) \( \Rightarrow g \) is a cusp form. Apply this to

see \( \overline{h}_k \) is a cusp form. \# since \( a(0) \neq 0 \).

2. Use the \( g \)-exp. principle (Deligne - Rapoport): \( \overline{h}_k \)

has Fourier coeff. in \( \mathbb{Q} \). If it were modular of

\( \text{at} \ d/2 \) \( \Rightarrow \) then the Fourier coeff. of the cusp 0 is in \( \mathbb{Q} \)

by this principle. \# to \( \frac{W_k}{|\det A|} \) by Lemma 2.

**Theorem 2:** Suppose \( d = 2 \) and \( k \) is an integral curve, i.e.

\( \omega_1, \omega_2 \in \mathbb{Z}^2 \). Then \( \overline{h}_{k, \mathbb{Z}^2} \) is not modular of \( \text{at} \ 1 \).
Proof: In most cases, one can use the g-exp principle:

\[ \omega \in \mathbb{Z}^{\frac{d}{2}}, \text{ suppose } \omega \neq 0 \quad \Rightarrow \quad Q = 2\pi \frac{d}{2} \quad \text{with} \]

\[ g \in \mathbb{Z}, \, d \neq 0. \quad \text{Also, } \quad \cos \theta = \frac{<\omega, \omega>}{\|\omega\|} = \frac{n}{\sqrt{m}} \]

(for \( n, m \in \mathbb{Z} \), \( m > 0 \), \( b \subset \mathbb{W}, \mathbb{W} \in \mathbb{Z}^{2} \))

\[ \Rightarrow \quad \cos\left(2\pi \frac{c}{d}\right) = \frac{n}{\sqrt{m}} \quad \text{is rational or quadratic.} \]

\[ \downarrow \]

Cyclotomic \( \mathfrak{p} \text{nd} \# \).

3) Modularity:

Theorem 3: Let \( V \) be the Weyl chamber of a finite \( \text{W} \) reflection group attached to one of the root systems \( A_n, B_n, C_n, D_n \).

Let \( \Phi^\text{W} \) be the corresponding root lattice. Then \( \Phi^\text{W} \) must lie in the graded ring of modular forms.

Proof: "Morally"

\[ \left( \oplus \Phi_{\text{nd}}, |d| = N \right). \]

Induction on \( d \).