MTHSC 102 Section 1.1 – Models and Functions

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Note

We will typically represent related data in four ways or from four viewpoints.

- Numerically (using a chart or table of data)
- Graphically (using a scatter plot or continuous graph)
- Verbally (using a word description)
- Algebraically (using a mathematical model).

Definition

The process of translating a real-world problem into a usable equation is called mathematical modeling. The resulting equation along with a description of the variables involved is referred to as a mathematical model.

Example

Suppose that we wish to fill in a low swampy area with 120 cubic yards of dirt. Suppose that we will use a truck which is capable of moving 12 cubic yards of soil per load and that we can deliver 1 load of soil per hour around the clock until the job is complete. The amount of soil moved after t hours is given by the following chart.

Elapsed time (hours)	Amount of Soil
1	12
2	24
4	48
6	72
8	96
10	120

Display this data graphically. Write a model for the amount of soil moved in terms of elapsed hours.

Definition

A function is a rule that assigns exactly one output value to each possible input value.

Example

If the price of gas is fixed at \$3.93 per gallon and g is the number of gallons that we wish to purchase, then the cost C will be a function of g, because for a given value of g, say $g = 10$ gallons there is only one possible cost that we would expect to pay, namely \$39.30.

We might represent the function of the previous example by

- The chart.

gallons	Cost
1	$ 3.93
4	$15.72
7	$ 27.51
10	$ 39.30

- A graph (draw the graph).
- An input/output diagram.

Note

The graph in this example is an example of a continuous graph because it can be drawn without lifting our pencil.

Determining the output of a function is dependent on the presentation of the function.

- Numerical (-i.e. chart or table): Read off the output.
- Graphical: Draw a vertical line which intersects the x-axis at the input value. Draw a horizontal line which meets the graph at the same place as the vertical line you just drew. The point of intersection of the horizontal line with the y-axis is the function output.
- Algebraic: If we have an equation relating the input variable to the output variable, we simply substitute the input for the input variable and calculate the value of the output variable.

Note (Vertical Line Test)

Suppose that we have a graph of data which are related in some way. If we can draw a vertical line which intersects the graph in 2 or more places then the variable whose values are plotted along the y-axis **cannot** be expressed as a function of the variable whose values are plotted along the x-axis.

Note

We have discussed three different concepts which are important to distinguish.

- Model: Always refers to an equation together with a description of the variables involved in the equation including units of measure.
- Equation: refers to a mathematical formula in one or more variables.
- Function: A rule which assigns to each possible input a unique output.

Note

Not all equations are functions. For example $x^2 + y^2 = 1$ is an equation but neither x nor y can be expressed as a function of the other.
If we are given the equation $y^2 - x = 2$, we can express x as a function of y but not the other way around.

Interpreting Model Output

In order to understand a mathematical model, it is important to identify and understand the units of measure involved.

Example

The value of a certain piece of property between the years 1980 and 2007 is given by the model

$$v(t) = 2.7(1.083)^t \text{ thousands of dollars.}$$

where t is the number of years since the end of 1980.
Graph the value of the property from 1980 to 2007 (-i.e. $0 \leq t \leq 27$).
Make a table of some of the values.
Describe the input and output variables. Be sure to discuss the units of measure involved.
What was the land value at the end of 2002?
When was the land worth \$ 18,000?

Before describing combination of functions and the meaning of such combinations, here are some typical variables which arise in business and their relationships.

Business Term	Description
Fixed Costs	Costs that occur regardless of the number of items produces
Variable Costs	Productions costs
Total Costs	Fixed Cost + Variable Costs
Average Cost	$\frac{\text{total cost}}{\text{\# items produced}}$
Revenue	Amount of income the business recieves
	Revenue = Profit + Total Cost
	Revenue = Price × Demand
Cost	Same as total cost
Profit	Profit = Revenue - Cost
Break-even point	Point where Cost = Revenue

NOTE

Functions can be constructed using addition and subtraction if

- The inputs of the two functions have the same description and units.
- The outputs of the two functions have the same units AND if there is a way to describe the resulting output if the functions are added or subtracted.
- In many examples the given functions may fail one of the above criteria, but can be adjusted so that the criteria are satisfied.

Example

Consider the total cost of a dairy business during a certain month. The fixed costs for the dairy for that month are \$ 21,000 and the variable cost incurred on day d of the month in question is given by $v(d) = -0.3d^2 + 6d + 250$ where $0 \leq d \leq 30$. Express the total cost of operations on day d of the month.

Functions can be constructed using multiplication or division if

- The inputs for both functions have the same description and units.
- The output units for the 2 functions are well defined when they are multiplied or divided.
- There is a meaningful description of the resulting output when the functions have been multiplied or divided.

Example

Again consider the dairy business from before. Suppose that the price of milk on day d is given by $M(d) = 0.019d + 1.85$ dollars per gallon ($0 \leq d \leq 30$) and that $S(d) = 1.7 + 0.4(0.82^d)$ thousand gallons of milk were sold on day d.
Draw separate input/output diagrams for M and S.
Determine if M and S are compatible for multiplication.
What is the resulting output unit of measure for $(M \cdot S)(d)$.
Draw an input/output diagram for the multiplication function.
Write a function for daily revenue from milk sales?

Example

Continuing our example from before, suppose that we know that the total production on day d was $Q(d) = -1.5d^2 + 32d + 1803$ gallons of milk and that the total cost incurred on day d was $C(d) = -0.31d^2 + 6.2d + 1035$ dollars where again $0 \leq d \leq 30$. What is the average cost of production of a gallon of milk on each day?

Example

Suppose that the total cost to produce g gallons of milk during a particular month is $K(g) = 20000 + 0.19g$ dollars.
Suppose also that the average price of milk during the month is \$ 2.25. Then the revenue from g gallons of milk can be modeled by $T(g) = 2.25g$ dollars.
Graph $K(g)$ and $T(g)$.
Write a function for the profit for the month if g gallons are produced and graph this function.
How much milk needs to be produced for the dairy to break even for the month?

Definition

Given two functions f and g we can form the composition function $g \circ f(x) = g(f(x))$ if the outputs of f can always be used as inputs for g. It must be the case also that the unit measure of the outputs of f are identical to the input units of g.

Example

Consider the descriptions for the following functions.
$C(p) =$ parts per million of contamination in a lake when the population of the surrounding community is p people.
$p(t) =$ the population in thousands of people of the lakeside community in year t.
Draw an input/output diagram for the composition function that gives the contamination in a lake as a function of time.
Suppose that $C(p) = \sqrt{p}$ parts per million and that $p(t) = 0.4t^2 + 2.5$ thousand people t years after 1980 $(t \geq 0)$.
Write a function that gives the lake contamination as a function of time.