§ 3.10 Linear Approximation & Differentials

Example: Approximate $\sqrt{5}$.

**Note:** We know $\sqrt{4} = 2$. How can this help?

**Idea:**

Note: If $b$ is close to $a$ then $\Delta b$ is close to $f(b)$

Thus, $f(b) \approx \Delta b = f'(a)(b-a) + f(a)$
Now take \( f(x) = \sqrt{x} \)
\[ \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} \]

Note that on the above graph it appears that \( f(5) \) is very close to \( f(5) \) (within 0.02).
So, to approximate $\sqrt{5}$ we use the tangent line to \( f(x) = \sqrt{x} \) at \((4, 2)\).

\[
\sqrt{5} = f(5) \approx L(5) = f'(4) (x-4) + f(4)
\]

\textbf{Note:} \hspace{1cm} f(4) = \sqrt{4} = 2

\[
f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

So

\[
\sqrt{5} \approx \frac{1}{4} (5-4) + 2 = 2.25.
\]

(Your calculator might give $\sqrt{5} \approx 2.236067977$)

**Definition:** We call the function

\[
L(x) = f(a) + f'(a)(x-a)
\]

the linearization of \( f \) at \( a \).
Compute the linearization of \( f(x) = \sqrt{x + 6} \) at \( a = 3 \).

Compare the values of \( L(x) \) & \( f(x) \) near 3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( L(x) )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example: For what values of $x$ is the linear approx. 

$$\sqrt{x+6} \approx 3 + \frac{1}{6}(x-3)$$ 

accurate to within 0.5?

Solve

$$|\sqrt{x+6} - (3 + \frac{1}{6}(x-3))| < 0.5$$

OR equivalently

$$\frac{1}{6}x + \frac{5}{3} - 1.5 < \sqrt{x+6} < \left(\frac{1}{6}x + \frac{5}{3}\right) + 0.5$$

$$\frac{1}{6}x + 1.2 < \sqrt{x+6} < \frac{1}{6}x + 3$$

$$\frac{1}{36}x^2 + \frac{2}{3}x + 4 < x + 6 < \frac{1}{36}x^2 + x + 9$$

$$\frac{1}{36}x^2 - \frac{1}{3}x - 2 < 0 \quad \& \quad 0 < \frac{1}{36}x^2 + 3$$

$$x^2 - 12x - 72 < 0$$

$$(x-6)^2 - 36 - 72 < 0$$

$$(x-6)^2 < 108$$

$$|x-6| < \sqrt{108}$$

$$6 - \sqrt{108} < x < 6 + \sqrt{108}$$

See p. 207 for a geometric interpretation.
Differentials:

Definition: The differential \( \mathrm{d}x \) is an independent variable (like \( x \)) and we define \( \mathrm{d}y \) as

\[
\mathrm{d}y = f'(x) \, \mathrm{d}x
\]

So,

\[
\Delta y \approx \mathrm{d}y
\]

Note: \( \mathrm{d}y \) depends on the values of \( x \) and \( \mathrm{d}x \).
Example: The radius of a sphere is measured to be 21 cm with a possible error of at most 0.05 cm. What is the maximum error in using this measure for the radius to compute the volume.