Chapter 5.3 The Fundamental Theorem of Calculus.

**Accumulation functions:**

Suppose that $f$ is cont. on $(a, \infty)$ then we define a function $g(x)$ as

$$g(x) = \int_a^x f(t) \, dt$$

*eg:*

$$y = f(t)$$

Put $g(0) = \int_0^x f(t) \, dt$

- $g(0) = \int_0^1 f(t) \, dt = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$
- $g(2) = \int_0^2 f(t) \, dt = \frac{1}{2} (2) \times 1 = 1$
- $g(3) = \int_0^3 f(t) \, dt = \int_0^2 f(t) \, dt + \int_2^3 f(t) \, dt$
  $$= 1 - \frac{1}{2} (1) \times 1 = \frac{1}{2}$$
- $g(4) = \int_0^4 f(t) \, dt = \int_0^2 f(t) \, dt + \int_2^4 f(t) \, dt$
  $$= 1 - \frac{1}{2} (2) \times 1 = 0$$

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Sketch the graph of \( g(x) = \int_0^x f(t) \, dt \)

\[ y = f(x) \]

**Note:**
- \( g \) is increasing on ______
- \( g \) is decreasing on ______
- \( g \) attains a max at \( x = \)______
- \( g \) attains a min at \( x = \)______

\[ y = g(x) \]
Take $f(t) = t$ and $a = 0$

Then

$$g(x) = \int_0^x f(t) \, dt$$

$$= \lim_{n \to \infty} \sum_{i=0}^{n} \frac{x}{n} \, \Delta t$$

$$= \lim_{n \to \infty} \sum_{i=0}^{n} \left( \frac{x}{n} \cdot \frac{x}{n} \right)$$

$$= \lim_{n \to \infty} \sum_{i=0}^{n} \frac{x^2}{n^2}$$

$$= \lim_{n \to \infty} \frac{x^2}{n^2} \sum_{i=0}^{n} i$$

$$= \lim_{n \to \infty} \frac{x^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{x^2}{n^2} \cdot \frac{n^2 + n}{2n^2}$$

$$= \frac{x^2}{2} \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2}$$

$$= \frac{1}{2} x^2$$

Note: $g'(x) = x = f(x)$. 
In general,

\[ y = f(x) \]

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\text{Note:} \quad g(x+h) - g(x) = \text{Area of the above rectangle}
\quad = f(x) \cdot h
\quad \Rightarrow \quad g(x+h) - g(x) \cdot \frac{1}{h} = f(x)
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So, per hops,

\[ g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \]
Fundamental Theorem (Part I):

If \( f \) is cont. on \([a, b]\) then the function

\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is cont. on \([a, b]\) and differentiable on \((a, b)\) and

\[
g'(x) = f(x)
\]

Alt:

\[
\frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x)
\]

Ex:

Let \( g(x) = \int_0^x \sqrt{1+t^2} \, dt \)

Then

\[
g'(x) =
\]
Example: Compute
\[
\frac{d}{dx} \left( \int_1^x \sec(t) \, dt \right)
\]

(\text{Hint: Don't forget the chain rule!})

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**Fundamental Theorem (Part 2),**

If \( f \) is cont. on \([a,b]\) then
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F(x) \) is any antiderivative of \( f(x) \) on \([a,b]\), that is \( F'(x) = f(x) \).

\text{Sketch of Proof, (see your book p. 345)}

Let \( g(x) = \int_a^x f(t) \, dt \). Then
\[
F(x) = g(x) + C
\]

Thus
\[
F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) - g(a)
\]
\[
= g(b) = \int_a^b f(t) \, dt
\]
Eg. Evaluate
\[ \int_{-2}^{1} x^2 \, dx \]

\[ F(x) \bigg|_{a}^{b} = F(b) - F(a) \]

Eg. \[ x^2 \bigg|_{1}^{3} = 3^2 - 1^2 = 9 - 1 = 8. \]

Eg. Find the area under the cosine curve from \( 0 \) to 1, where \( 0 \leq b \leq \frac{\pi}{2} \).
BAD Use of Fund Thm:

\[-\int_{-1}^{1} \frac{1}{x^2} \, dx = \left. -\frac{1}{x} \right|_{-1}^{1} = -\frac{1}{1} - \frac{1}{-1} = -1 - 1 = -2\]

What is wrong with this example?
Fund. Thm.:

1) If \( g(x) = \int_a^x f(t) \, dt \), then
   \[ g'(x) = f(x) \]

2) If \( F'(x) = f(x) \) then
   \[ \int_a^b f(x) \, dx = F(b) - F(a) \]

\[
\left( \text{If } F'(x) = f(x) \text{ then } \int_a^x f(t) \, dt = F(x) - \frac{F(a)}{u} \right)_{\text{constant}}
\]