MAT 106
Quiz #10
30 September 2004

Name:
You may not use your notes. Please show all of your work. An answer without justification will receive little credit.

(1) Identify the absolute extrema of the function \( f(x) = x^3 - 9x^2 + 15x + 2 \) on the interval \([0, 6]\).

First we compute the derivative of \( f(x) \).

\[ f'(x) = 3x^2 - 18x + 15 = 3(x - 1)(x - 5). \]

Since \( f \) and \( f' \) are defined on all points in \([0, 6]\) and since the zeros of \( f'(x) \) are 1 and 5, the critical numbers of \( f(x) \) are 1 and 5. Now we must evaluate \( f \) at 0, 1, 5 and 6.

\[
\begin{align*}
  f(0) &= 2, \\
  f(1) &= 9, \\
  f(5) &= -23 \quad \text{and} \\
  f(6) &= -16.
\end{align*}
\]

Since all local extrema must be attained at the critical numbers and since we have also evaluated \( f \) at the endpoints of \([0, 6]\), we can see that the absolute maximum value of \( f \) on \([0, 6]\) is 9 and this value is attained at \( x = 1 \). Also, we can see that the absolute minimum value of \( f \) on \([0, 6]\) is -23 and this value is attained at \( x = 5 \).

(2) Prove that \( g(x) = x^5 + 7x^3 + 2x + 7 \) has exactly one real root.

First, we note that \( g \) is continuous and differentiable everywhere because \( g \) is a polynomial. Next, we note that \( g(-1) = -3 \) and that \( g(0) = 7 \). Since \(-3 = g(-1) < 0 < g(0) = 7\), the Intermediate Value Theorem, states that there exists a number \( c \) with \(-1 < c < 0\) such that \( g(c) = 0 \). Thus \( g \) has at least one root.

Now, suppose there were two roots, say at \( d \) and \( e \) where \( d < e \) (one of these may be the same as \( c \) above). Since \( g \) is a polynomial, we have

a.) \( g \) is continuous on \([d, e]\),

b.) \( g \) is differentiable on \((d, e)\) and

c.) \( g(d) = 0 = g(e) \).

Thus Rolle’s theorem states that there is a number \( b \in (d, e) \) such that \( g'(b) = 0 \). That is, we have proved that if \( g \) has two roots, then there must be a number \( b \) such that \( g'(b) = 0 \). However, \( g'(x) = 5x^4 + 21x^2 + 2 \) which is always greater than 0. Thus \( g \) cannot have a second root.