MAT 106
Quiz #10
30 September 2004

Name:____
You may not use your notes. Please show all of your work. An answer without justification will receive little credit.

(1) Identify the absolute extrema of the function $f(x) = x^3 - 9x^2 + 15x + 2$ on the interval $[0, 6]$.

First we comput the derivative of $f(x)$.

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5).$$

Since f and f' are defined on all points in $[0, 6]$ and since the zeros of $f'(x)$ are 1 and 5, the critical numbers of $f(x)$ are 1 and 5. Now we must evaluate f at 0,1,5 and 6.

$$\begin{aligned} f(0) &= 2, \\ f(1) &= 9, \\ f(5) &= -23 \quad \text{and} \\ f(6) &= -16. \end{aligned}$$

Since all local extrema must be attained at the critical numbers and since we have also evaluated f at the endpoints of $[0, 6]$, we can see that the **absolute maximum value** of f on $[0, 6]$ is 9 and this value is attained at $x = 1$. Also, we can see that the **absolute minimum value** of f on $[0, 6]$ is -23 and this value is attained at $x = 5$.

(2) Prove that $g(x) = x^5 + 7x^3 + 2x + 7$ has exactly one real root.

First, we note that g is continuous and differentiable everywhere because g is a polynomial. Next, we note that $g(-1) = -3$ and that $g(0) = 7$. Since $-3 = g(-1) < 0 < g(0) = 7$, the Intermediate Value Theorem, states that there exists a number c with $-1 < c < 0$ such that $g(c) = 0$. Thus g has at least one root.

Now, suppose there were two roots, say at d and e where $d < e$ (one of these may be the same as c above). Since g is a polynomial, we have
a.) g is continuous on $[d, e]$,
b.) g is differentiable on (d, e) and
c.) $g(d) = 0 = g(e)$.
Thus Rolle's theorem states that there is a number $b \in (d, e)$ such that $g'(b) = 0$. That is, we have proved that if g has two roots, then there must be a number b such that $g'(b) = 0$. However, $g'(x) = 5x^4 + 21x^2 + 2$ which is always greater than 0. Thus g cannot have a second root.