§ 2.8 Factorial

- **Example:** The number of length n lists from $1, 2, \ldots, n$ without repeats is

$$(n)_n = n(n-1)(n-2)\cdot\ldots\cdot 2\cdot 1$$

- **Def:** If $n \geq 1$, then we define.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

We also define $0! = 1$.

- **Example:**

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3! = 24.$$

- **Product Notation:**

- **Definition:**

$$a_1 \times a_2 \times \ldots \times a_k = \prod_{i=1}^{k} a_i$$

- **Example:** $\displaystyle\prod_{k=1}^{n} k = n!$

- **Example:** $\displaystyle\prod_{k=1}^{3} (2k+3) =$

Example:

$$\prod_{i=1}^{3} 2 =$$

Example:

$$\frac{103!}{101!} =$$

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§2.9 Sets I.

- Definition: A set is a **repetition free** unordered collection of objects called elements.

- Example:

(1) $\{1,2,3\}$, $\{1,5,7\}$, $\{A, B, 2, \gamma, z\}$ are sets.

(2) Let $S = \{1,2,3\}$. Then

$1 \in S$; $2 \in S$; $5 \notin S$

(3) $\emptyset = \{\}$ (the empty set)

- **Definition:** If A is a set, we denote the number of elements in A (or the cardinality of A) by $|A|$ or $\#A$.

- Another way to describe a set is:

$$\{\text{variables} : \text{conditions}\}$$

- **Example:**

$$\mathcal{E} = \{x \in \mathbb{Z} : 2 \mid x\} \quad \text{(even integers)}$$

$$\mathcal{O} = \{x \in \mathbb{Z} : 2 \nmid x\}$$

- **Definition:** Two sets are equal provided they have precisely the same elements.

- **Definition:** (Subset) Suppose A and B are sets. We say that A is a subset of B ($A \subseteq B$) provided that each element of A is also an element of B.

- **Example:**

$$S = \{1, 3, 5, 7\}$$
$$T = \{2, 4, 6, 8\}$$
$$R = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$W = \{z \in \mathbb{Z} : 1 \le z \le 8\}$$

S ___ R, $\quad$ R ___ W

T ___ R, $\quad$ R ___ T.

Note: If $S = \{x, y, z\}$ then

$x \in S$; $\{x\} \subseteq S$.

~~Proposition~~

Proof Template 6:

To show $A \subseteq B$:

Let $x \in A$.

$\vdots$

Therefore $x \in B$. Thus $A \subseteq B$.

Prop 9.3: Let x be anything and let A be a set. Then, $x \in A \iff \{x\} \subseteq A$.

Pf:

($\Rightarrow$): Suppose that $x \in A$.

Note that the only element of $\{x\}$ is x, and $x \in A$.

Thus, $\{x\} \subseteq A$.

($\Leftarrow$): Now suppose that $\{x\} \subseteq A$.

Thus $x \in A$.

- **Proof Template 5:** Let A, B be sets.

  To show $A = B$:

  1) Show $A \subseteq B$.
  2) Show $B \subseteq A$.

- **Prop 9.1:** Let

$$E = \{x \in \mathbb{Z} : x \text{ is even}\}$$
$$F = \{z \in \mathbb{Z} : z = a+b \text{ where } a, b \text{ are odd}\}.$$

Then $E = F$.

Proof:

($\subseteq$): First we will show that $E \subseteq F$.
Suppose that $x \in E$.

Therefore ___ + ___ $= x$ and ___ and ___ are odd.
Thus, $x \in F$, so, $E \subseteq F$.

($\supseteq$): Suppose that $x \in F$.

Therefore $x \in E$. So $F \subseteq E$.
Since $E \subseteq F$ and $F \subseteq E$, $E = F$.

- **Def:** (Pythagorean Triple)

  a list of 3 integers (a,b,c) is called a Pythagorean triple provided $a^2+b^2=c^2$.

- **Example:**

  $(3,4,5)$ is a Pythagorean triple.

- **Prop 9.5:** Let P be the set of Pythagorean triples, that is let $P=\{(a,b,c) : a,b,c\in\mathbb{Z};\ a^2+b^2=c^2\}$.

  Let $T=\{(p,q,r) : p=x^2-y^2;\ q=2xy;\ r=x^2+y^2;\ x,y\in\mathbb{Z}\}$.

  Then $T\subseteq P$.

- **Proof:**

  Let $(w,x,y)\in T$.

  Then $(w,x,y)\in P$. Thus, $T\subseteq P$.