§11 Set Operations:

Definition: Suppose that A and B are sets. Then we define the <u>union</u> $A \cup B$ of A and B and the <u>intersection</u> $A \cap B$ as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Example:

$$A = \{a, b, c, d\}$$

$$B = \{c, d, e, f\}$$

$$A \cup B = \{ \qquad \qquad \qquad \}.$$

$$A \cap B = \{ \qquad \}.$$

Theorem 11.3:

1) $A \cup B = B \cup A$; $A \cap B = B \cap A$.
2) $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$.
3) $A \cup \emptyset = A$; $A \cap \emptyset = \emptyset$.
4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

We will show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

(Note: We will use Thm 6.2 $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.)

Method 1:

($\subseteq$): Let $x \in A \cup (B \cap C)$

Then

Thus, $x \in (A \cup B) \cap (A \cup C)$ and therefore

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C).$$

($\supseteq$): Let $x \in (A \cup B) \cap (A \cup C)$.

Then

So, $x \in A \cup (B \cap C)$ and therefore

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

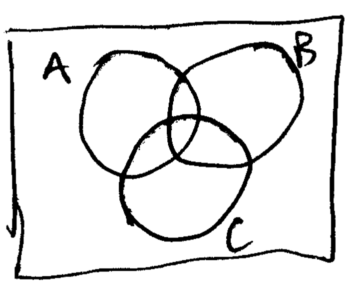
Alternative Method:

$$A \cup (B \cap C) = \{ x : x \in A \vee x \in B \cap C \}$$

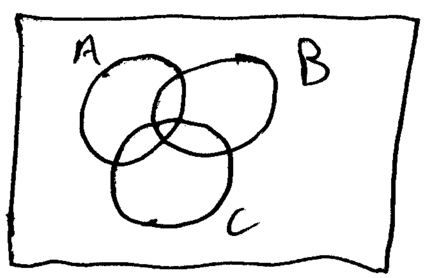
$$= \{$$

Venn Diagrams:

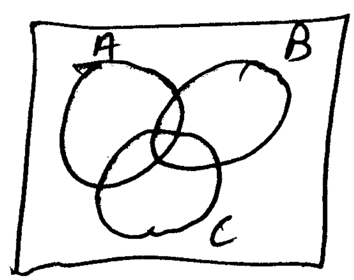
Diagrams that are useful in visualizing which things are true. However, a Venn Diagram is <u>**NOT**</u> a proof.



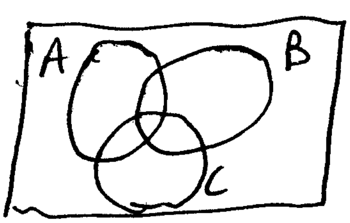
A



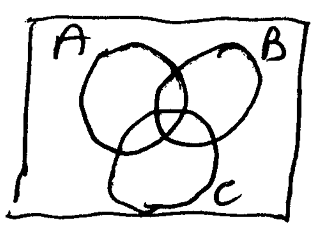
$B \cup C$



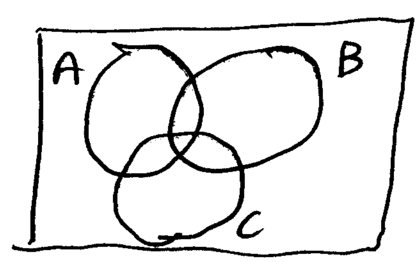
$A \cap (B \cup C)$



$(A \cap B)$



$(A \cap C)$



$(A \cap B) \cup (A \cap C)$

Prop. 11.4:

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Cor: $|A \cup B| = |A| + |B| - |A \cap B|.$

Proof (of Prop. 11.4):

We will give a combinatorial proof that $|A| + |B| = |A \cup B| + |A \cap B|$.

First, label each element of A with an "a".

Label each element of B with a "B".

How many labels were used?

I.

II.

Thus, $|A| + |B| = \#\text{labels} = |A \cup B| + |A \cap B|$ $\blacksquare$

Proof Template 9:

To prove an equation of the form:

$$LHS = RHS$$

1. Pose a counting question: "In how many ways...?"
2. Argue that LHS is the answer.
3. On the other hand, argue that RHS is the right answer.
4. Conclude that

$$LHS = \text{Answer} = RHS.$$

Note: Finding the right question is the difficult part.
To get started, ask yourself what is the LHS (or RHS) counting.

Example: How many integers $1 \le x \le 1000$ are divisible by 2 or 5?

Definition 11.6:

1) Let A and B be sets. We say that A and B are <u>disjoint</u> provided that $A \cap B = \emptyset$.

2) Let $A_1, \ldots, A_n$ be a collection of sets. This collection is said to be <u>pairwise disjoint</u> provided that whenever $i \neq j$, $A_i \cap A_j = \emptyset$.

· <u>Example:</u>

$A = \{1, 2, 3\}$
$B = \{4, 5, 6\}$
$C = \{7, 8, 9\}$

$A \cap B =$

$A \cap C =$

$B \cap C =$

<u>Cor 11.8:</u> Let A and B be sets. If A and B are disjoint then $|A \cup B| = |A| + |B|$.

<u>Proof:</u>

- **Cor:** Suppose that $A_1, \dots, A_n$ is a pairwise disjoint collection of sets. Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = |A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| .$$

- **Definition:** Let A and B be sets. Then we define their difference $A-B$ and their symmetric difference $A \Delta B$ by

$$A - B = \{ x \in A : x \notin B \}$$

$$A \Delta B = (A-B) \cup (B-A)$$

- **Example:** $A = \{1, 2, 3, 4\}$

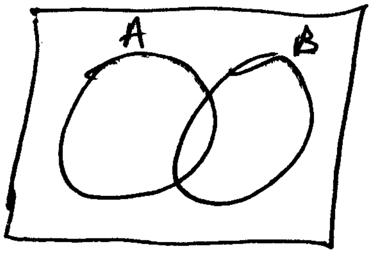
$B = \{1, 4, 7, 9\}$

$A - B = \{ \quad \}$

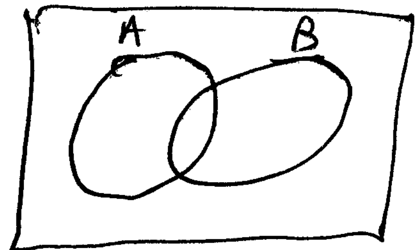
$B - A = \{ \quad \}$

$A \Delta B = \{ \quad \}.$

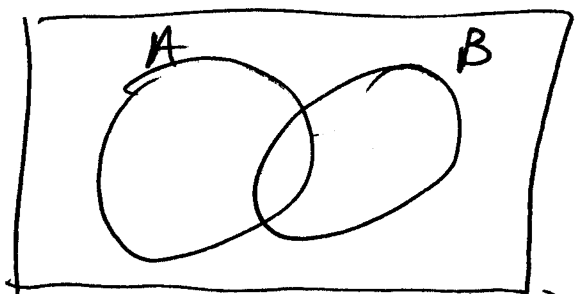
Let's draw a picture of symmetric difference:



$A - B$



$B - A$



$A \Delta B = (A-B) \cup (B-A)$

<u>Prop 11.11:</u> Let A and B be sets. Then

$$A \Delta B = (A \cup B) - (A \cap B)$$

<u>Proof:</u> First, we will show that $A \Delta B \subseteq (A \cup B) - (A \cap B)$

$(\subseteq)$: Let $x \in A \Delta B = (A-B) \cup (B-A)$.

Then $x \in A - B$ <u>or</u> $x \in B - A$.

<u>Case 1:</u> Suppose $x \in A - B$. Then,

<u>Case 2:</u> Suppose $x \in B - A$. Then,

Since in either case,

$$x \in (A \cup B) - (A \cap B),$$

$$A \Delta B \subseteq (A \cup B) - (A \cap B).$$

($\supseteq$): Now, we will show that

$$(A \cup B) - (A \cap B) \subseteq A \Delta B.$$

Let $x \in (A \cup B) - (A \cap B)$.

So again there two cases: $x \in A$ or $x \in B$.

Case 1: Suppose that $x \in A$.

Case 2: Suppose that $x \in B$.

So in either case $x \in (A - B) \cup (B - A) = A \Delta B$.

Thus, $A \Delta B \subseteq (A - B) \cup (B - A)$.

So, we have shown that

$$A \Delta B = (A \cup B) - (A \cap B).$$

Definition:

Let A and B be sets. Then the Cartesian product of A and B is defined as

$$A \times B = \{(a,b) : a \in A;\ b \in B\}$$

Example: $A = \{1,2,3\}$ $B = \{x, y\}$

$A \times B = \{ \qquad \}$

$B \times A = \{ \qquad \}$

Prop 11.15: Let A and B be finite sets. Then, $|A \times B| = |A|\,|B|$.

Proof: