Definitions: Suppose that \( A \) and \( B \) are sets.

Then we define the union \( \cup \) of \( A \) and \( B \) and the intersection \( \cap \)

\[ A \cup B = \{ x : x \in A \text{ or } x \in B \} \]

\[ A \cap B = \{ x : x \in A \text{ and } x \in B \} \]

Examples:

\[ A = \{ a, b, c, d \} \]

\[ B = \{ c, d, e, f \} \]

\[ A \cup B = \{ a, b, c, d, e, f \} \]

\[ A \cap B = \{ c, d \} \]

Theorem 11.3:

1. \( A \cup B = B \cup A \); \( A \cap B = B \cap A \).
2. \( (A \cup B) \cap C = A \cup (B \cap C) \); \( (A \cap B) \cap C = A \cap (B \cap C) \).
3. \( A \cup \emptyset = A \); \( A \cap \emptyset = \emptyset \).
4. \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \); \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \).
Proof:
We will show that
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]
(Note: We will use Thm 6.2 \( x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \).

\( \leq \): Let \( x \in A \cup (B \cap C) \)

Then

Thus, \( x \in (A \cup B) \cap (A \cup C) \) and therefore
\[ A \cup (B \cap C) \leq (A \cup B) \cap (A \cup C). \]

\( \geq \): Let \( x \in (A \cup B) \cap (A \cup C) \).

Then

So, \( x \in A \cup (B \cap C) \) and therefore
\[ (A \cup B) \cap (A \cup C) \leq A \cup (B \cap C). \]

Thus,
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]
Alternative Method:

\[ A \lor (B \land C) = \exists x : x \in A \lor \exists x : x \in B \land C \]

\[ = \exists \]

Venn Diagrams:

Diagrams that are useful in visualizing which things are true. However, a Venn Diagram is **NOT** a proof.
Prop. 11.4:

\[ |A| + |B| = |A \cup B| + |A \cap B| \]

Cor: \[ |A \cup B| = |A| + |B| - |A \cap B| \]

Proof (of Prop. 11.4):

We will give a combinatorial proof that \[ |A| + |B| = |A \cup B| + |A \cap B| \].

First, label each element of A with an "A".

Label each element of B with a "B".

How many labels were used?

Thus, \[ |A| + |B| = \# \text{labels} = |A \cup B| + |A \cap B| \]
Proof Template 9:

To prove an equation of the form:

\[ \text{LHS} = \text{RHS} \]

1. Pose a counting question: "In how many ways...?"

2. Argue that LHS is the answer.

3. On the other hand, argue that RHS is the right answer.

4. Conclude that

\[ \text{LHS} = \text{Answer} = \text{RHS} \]

Note: Finding the right question is the difficult part.

To get started, ask yourself what is the LHS (or RHS) counting.

Example: How many integers 1 ≤ x ≤ 1000 are divisible by 2 or 5?
**Definition 11.6:**

1. Let $A$ and $B$ be sets. We say that $A$ and $B$ are **disjoint** provided that $A \cap B = \emptyset$.

2. Let $A_1, \ldots, A_n$ be a collection of sets. This collection is said to be **pairwise disjoint** provided that whenever $i \neq j$, $A_i \cap A_j = \emptyset$.

**Example:**

$A = \{1, 2, 3\}$

$B = \{4, 5, 6\}$

$C = \{7, 8, 9\}$

$A \cap B = \emptyset$

$A \cap C = \emptyset$

$B \cap C = \emptyset$

**Corollary 11.8:** Let $A$ and $B$ be sets. If $A$ and $B$ are disjoint then $|A \cup B| = |A| + |B|$.

**Proof:**
Corollary: Suppose that $A_1, \ldots, A_n$ is a pairwise disjoint collection of sets. Then
\[
\left| \bigcup_{i=1}^{n} A_i \right| = |A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|.
\]

Definition: Let $A$ and $B$ be sets. Then we define their difference $A - B$ and their symmetric difference $A \Delta B$ by
\[
A - B = \{ x \in A : x \not\in B \},
\]
\[
A \Delta B = (A - B) \cup (B - A).
\]

Example:
\[
A = \{1, 2, 3, 4, 5\},
\]
\[
B = \{1, 4, 7, 9\},
\]
\[
A - B = \{2, 3\},
\]
\[
B - A = \{7, 9\},
\]
\[
A \Delta B = \{2, 3, 7, 9\}.
\]
Let's draw a picture of symmetric difference:

\[ A \Delta B = (A \cap B) \cup (B \cup A) \]

**Proposition:** Let \( A \) and \( B \) be sets. Then

\[ A \Delta B = (A \cup B) - (A \cap B) \]

**Proof:** First, we will show that \( A \Delta B \subseteq (A \cup B) - (A \cap B) \).

Let \( x \in A \Delta B \), then \( x \in (A \not\subseteq B) \cup (B \not\subseteq A) \).

**Case 1:** Suppose \( x \in A - B \). Then,

**Case 2:** Suppose \( x \in B - A \). Then,
Since in either case,
\[ x \in (A \cup B) - (A \cap B), \]
\[ A \Delta B = (A \cup B) - (A \cap B). \]

(2): Now, we will show that
\[ (A \cup B) - (A \cap B) \subseteq A \Delta B. \]
Let \( x \in (A \cup B) - (A \cap B) \).
So again there are two cases: \( x \in A \) or \( x \in B \).

Case 1: Suppose that \( x \in A \),

Case 2: Suppose that \( x \in B \),

so in either case \( x \in (A - B) \cup (B - A) = A \Delta B \).
Thus,
\[ A \Delta B \subseteq (A - B) \cup (B - A). \]
So, we have shown that
\[ A \Delta B = (A \cup B) - (A \cap B). \]
Definition:

Let $A$ and $B$ be sets. Then the Cartesian product of $A$ and $B$ is defined as

$$A \times B = \{(a,b) : a \in A; b \in B\}$$

Example:

$A = \{1, 2, 3\}$, $B = \{x, y\}$

$A \times B = \{\}$

$B \times A = \{\}$

Proposition 15:

Let $A$ and $B$ be finite sets. Then, $|A \times B| = |A| \cdot |B|$. 

Proof: