Proposition 2.1:

Let \( n > 0 \). Then

\[ 2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1 \]

Notes:

1. The LHS counts the number of subsets of a \( k \)-element set for \( 0 \leq k \leq n-1 \) and adds them all up.

2. The RHS counts the number of subsets of \( \{1, \ldots, n \} \) minus one. So, it counts the number of nonempty subsets of \( \{1, 2, \ldots, n \} \), (there is only one empty set.)

How can we relate these things?

Examples:

<table>
<thead>
<tr>
<th>Largest element</th>
<th>Nonempty subsets of ( {1, 2, 3, \ldots } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<tr>
<td>3</td>
<td></td>
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</tbody>
</table>
Proof (Prop. 12.1):

Let $n > 0$ and let $N = \{1, 2, \ldots, n^2\}$.

How many nonempty subsets does $N$ have?

Answer 1:

We know that $\emptyset \subseteq N$ and $\emptyset \in 2^N$ and $|2^N| = 2^n$. Since $\emptyset$ is the only empty subset of $N$, the number of nonempty subsets is $|2^N - \{\emptyset\}| = 2^n - 1$.

Answer 2:

Consider the subsets of $N$ whose largest element is $j$. If $S$ is such a subset then $S - \{j^2\} \subseteq \{1, 2, \ldots, j - 3\}$.
Thus, there are $2^{j-1}$ such subsets.

So, the number of nonempty subsets of $N$ is

\[
\sum_{j=1}^{n} \text{(\# subsets whose largest element is $j$)}
\]

\[
= \sum_{j=1}^{n} 2^{j-1} = \sum_{k=0}^{n-1} 2^k
\]

Thus, $2^{n+1} = \text{Ans. 1} = \text{Ans. 2} = \sum_{k=0}^{n-1} 2^k$.
Def 13.1: A relation is a set of ordered pairs.

Example:

\[ R = \{ (0,0), (0,1), (1,2), (2,5) \} \]

Notation:

If \((x,y) \in R\) we say that "\(x\) is related to \(y\) (by \(R\))",
and we may write \(x \mathrel{R} y\).

Example:

The \(<\) relation can be thought of as the set

\[ < = \{ (-1,0), (-1,1), \ldots, (0,1), (0,2), (0,3), \ldots, (1,2) \} \]

Def 13.2: Let \(R\) be a relation and let \(A, B\) be sets.

1) We say that "\(R\) is a relation on \(A\)" provided that \(R \subseteq A \times A\).

2) We say that "\(R\) is a relation from \(A\) to \(B\)" provided that \(R \subseteq A \times B\).
Example:

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

$R = \{ (1,1), (2,2), (3,3) \}$

$S = \{ (1,2), (2,3) \}$

$T = \{ (2,4), (3,5) \}$

$U = \{ (4,1), (3,2), (5,3) \}$

$V = \{ (1,3), (2,5) \}$

Then

- $R$ is a relation
- $S$ is a relation
- $T$ is a relation
- $U$ is a relation
- $V$ is
Definition 13.1:

Let $R$ be a relation. The inverse of $R$, denoted $R^{-1}$, is the relation formed by reversing the order of all pairs in $R$.

(i.e. $R^{-1} = \{(y,x) : (x,y) \in R\}$)

Example:

$R = \{(1,1), (1,2), (1,3), (2,4), (4,2)\}$

$R^{-1} = \emptyset$

Note: If $R$ is a relation from $A$ to $B$, then $R^{-1}$ is a relation from $B$ to $A$.

Proposition 13.6.3: Let $R$ be a relation, then $(R^{-1})^{-1} = R$.

Proof:
(1): Let $(xy) \in (R^{-1})^{-1}$.

Then $(xy) \in R$. Thus, $(R^{-1})^{-1} \subseteq R$.

(2): Let $(xy) \in R$.

Then $(xy) \in (R^{-1})^{-1}$, Thus, $R = (R^{-1})^{-1}$.
**Definition 13.30:**

Let $R$ be a relation defined on a set $A$.

1. If $\forall x \in A$, $xRx$ then $R$ is reflexive.

2. If $\forall x \in A$, $x \not R x$ then $R$ is irreflexive.

3. If $\forall x,y \in A$, $xRy \implies yRx$ then $R$ is symmetric.
   (i.e., if $(x, y) \in R$, $(y, x) \in R$ then $R$ is symmetric.)

4. If $\forall x,y \in A$ $(xRy \land yRx) \implies x=y$ then $R$ is antisymmetric.
   (i.e., for $R$ to be antisymmetric, the following must be true:
   - If $x \neq y$ and $(x, y) \in R$ then $(y, x) \notin R$.
   - If $x \neq y$ and $(y, x) \in R$ then $(x, y) \notin R$.)

5. If $\forall x,y,z \in A$, $(xRy \land yRz) \implies xRz$ then $R$ is transitive.

**Example:** Let $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (3,3), (4,4)\}$

Ref. ——— Irref. ———
Sym ——— Antisym ———
Trans ———

$S = \{(1,1), (1,2), (2,1), (3,2), (2,3)\}$

Ref. ——— Irref. ———
Sym ——— Antisym ———
Trans ———
\[ T = \{ (1,2), (1,3), (2,3) \} \]

\[ U = \{ (1,1), (2,3), (3,1), (1,2) \} \]

Example:

1. \( < \) on \( \mathbb{Z} \)

\[ \text{Ref} \quad \text{Irref} \quad \text{Sym} \quad \text{Antisym} \quad \text{Trans} \]

2. \( \leq \) on \( \mathbb{Z} \)

\[ \text{Ref} \quad \text{Irref} \quad \text{Sym} \quad \text{Antisym} \quad \text{Trans} \]
Example: 1. on \( \mathbb{R}^n \):

- Rot,
- Irrot,
- Sym,
- Antisym,
- Trans,

2. on \( \mathbb{Z} \):

- Ref,
- Irref,
- Sym,
- Antisym,
- Trans,