§12 Combinatorial Proof.

- **Proposition 12.1:**

  Let $n > 0$. Then
  $$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1$$

- **Note:**

  1) The LHS counts the # of subsets of a k-element set for $0 \le k \le n-1$ and adds them all up.

  2) The RHS counts the number of subsets of $\{1, \ldots, n\}$ minus one. So, it counts the # of nonempty subsets of $\{1, 2, \ldots, n\}$. (There is only one empty set.)

How can we relate thes things?

- **Example:**

Largest element	Nonempty Subsets of $\{1,2,3\}$
1	
2	
3	

- Proof (Prop. 12.1):

Let $n \geq 0$ and let $N = \{1, 2, \dots, n\}$.
How many nonempty subsets does N have?

- Answer 1:

We know that $\emptyset \subseteq N$ and $\emptyset \in 2^N$ and $|2^N| = 2^n$. Since $\emptyset$ is the only empty subset of N, the number of nonempty subsets is $|2^N - \{\emptyset\}| = 2^n - 1$.

- Answer 2:

Consider the subsets of N whose largest element is j. If S is such a subset then $(S - \{j\}) \subseteq \{1, 2, \dots, j-1\}$
Thus, there are 2^{j-1} such subsets.
So, the # of nonempty subsets of N is

$$\sum_{j=1}^{n} (\text{\# subsets whose largest element is } j)$$

$$= \sum_{j=1}^{n} 2^{j-1} = \sum_{k=0}^{n-1} 2^k$$

Thus, $2^{n}-1 = \text{Ans. } 1 = \text{Ans. } 2 = \sum_{k=0}^{n-1} 2^k$ ∎

Def. 13.1:

A relation is a set of ordered pairs.

Example:

$$R = \{(0,0), (0,1), (1,2), (2,5)\}$$

Notation:

If $(x,y) \in R$ we say that "x is related to y (by R)" and we may write xRy.
If $(x,y) \notin R$, we may write $x \not R y$

Example:

The $<$ relation can be thought of as the set

$$< = \{\ldots (-1,0), (-1,1), \ldots, (0,1), (0,2), (0,3) \ldots (1,2) \ldots\}$$

Def. 13.2:

Let R be a relation and let A, B be sets.

1.) We say that "R is a relation on A" provided that $R \subseteq A \times A$.
2.) We say that "R is a relation from A to B" provided that $R \subseteq A \times B$.

Example:

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

$R = \{(1,1)\ (2,2)\ (3,3)\}$

$S = \{(1,2)\ (2,3)\ \}$

$T = \{(2,4), (3,5)\}$

$U = \{(4,1), (3,2), (5,3)\}$

$V = \{(17,32)\ (0,5)\ (1,3)\ (2,5)\}$

Then

R is a relation ______________

S is a relation ______________

T is a relation ______________

U is a relation ______________

V is ______________________

Definition 13.4:

Let R be a relation. The inverse of R denoted R^{-1} is the relation formed by reversing the order of all pairs in R.

(i.e. $R^{-1} = \{(y,x) : (x,y) \in R\}$)

Example:

$$R = \{(1,1)\ (1,2)\ (1,3)\ (2,4)\ (4,2)\}$$

$$R^{-1} = \{ \qquad \}$$

Note: If R is a relation from A to B then R^{-1} is a relation from B to A.

Proposition 13.6: Let R be a relation. Then $(R^{-1})^{-1} = R$.

Proof:

($\subseteq$): Let $(x,y) \in (R^{-1})^{-1}$.

Then $(x,y) \in R$. Thus, $(R^{-1})^{-1} \subseteq R$.

($\supseteq$): Let $(x,y) \in R$.

Then $(x,y) \in (R^{-1})^{-1}$. Thus, $R = (R^{-1})^{-1}$ ∎

- **Definition 13.7:**

Let R be a relation defined on a set A.

* 1.) If $\forall x \in A$, xRx then R is reflexive.

2.) If $\forall x \in A$, $x \not R x$ then R is irreflexive.

* 3.) If $\forall x,y \in A$, $xRy \Rightarrow yRx$ then R is symmetric.
(i.e. If $\forall (x,y) \in R$, $(y,x) \in R$ then R is symmetric.)

4.) If $\forall x,y \in A$ $(xRy \wedge yRx) \Rightarrow x = y$ then R is antisymmetric.
(i.e. For R to be antisymmetric, the following must be true
If $x \neq y$ and $(x,y) \in R$ then $(y,x) \notin R$.)

* 5.) If $\forall x,y,z \in R$, $(xRy \wedge yRz) \Rightarrow xRz$ then R is transitive.

Example: Let $A = \{1,2,3,4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

Ref. ______ Irref ______

Sym ______ Antisym ______

Trans ______

$S = \{(1,1),(1,2),(2,1),(3,2)\ (2,3)\}$

Ref. ______ Irref ______

Sym ______ Antisym ______

Trans ______

$T = \{(1,2),(1,3),(2,3)\}$

Ref ______ Irref ______

Sym ______ Antisym ______

Trans ______

$U = \{(1,1),(2,3),(3,1),(1,2)\}$

Ref ______ Irref ______

Sym ______ Antisym ______

Trans ______

Example:

1.) $<$ on $\mathbb{Z}$

Ref. ______ Irref. ______

Sym. ______ Antisym. ______

Trans. ______

2.) $\leq$ on $\mathbb{Z}$

Ref. ______ Irref. ______

Sym ______ Antisym ______

Trans. ______

Example: 1. $\mid$ on $\mathbb{N}$.

Ref. ______ Irref ______

Sym ______ Antisym ______

Trans ______

2. $\mid$ on $\mathbb{Z}$.

Ref. ______ Irref. ______

Sym ______ Antisym ______

Trans. ______