§14 Equivalence Relations.

- **Def:** Let R be a relation on a set A. We say that R is an <u>equivalence relation</u> provided that R is
  1. Reflexive
  2. Symmetric
  3. Transitive.

- <u>Example:</u> Let $\mathcal{S}$ be the set of all sets. Let R be defined as follows.
  If A and B are sets (i.e. $A, B \in \mathcal{S}$)
  then $A R B \iff |A| = |B|$
  (e.g. $\{1,2,3\} R \{a,b,c\}$; $\{a,b\} \not R \{x,y,z\}$)

<u>Reflexive:</u>

<u>Symmetric:</u>

<u>Transitive:</u>

- **Definition:** Let n be a positive integer. We say that integers x and y are congruent modulo n and write
$$x \equiv y \pmod{n}$$
provided that $n \mid (x-y)$.

- **Examples:** $3 \equiv 13 \pmod{5}$

$$10 \equiv 200 \pmod{19}$$

$$-5 \equiv 3 \pmod{4}.$$

- **Theorem 14.5:**
Let n be a positive integer. The $\equiv \pmod{n}$ relation is an equivalence relation on $\mathbb{Z}$.

- **Pf:** We must show that $= \pmod{n}$ is reflexive, symmetric and transitive.

Reflexive:

Symmetric:

Transitive:

- **Definition 14.6:**

Let R be an equivalence relation on a set A and let $a \in A$. The <u>equivalence class</u> of a, denoted $[a]$ is defined as

$$[a] = \{ x \in A : x \, R \, a \}$$

- **Example:**

Consider $\equiv \pmod 2$ on $\mathbb{Z}$,

$[1] = \{ \qquad \}$

$[2] = \{ \qquad \}$

$[3] = \{ \qquad \}$

- **Example:** Let $S = \{1, 2, 3\}$, Consider the "has the same size" relation on 2^S.

$[\emptyset] =$

$[\{1\}] =$

$[\{1,2\}] =$

$[\{1,2,3\}] =$

· **Prop 14.9:**
Let R be an equivalence relation on a set A, and let $a \in A$. Then $a \in [a]$.

· **Pf:**

· **Cor:** Let R be an equivalence relation on a set A.

1.> $[a] \neq \emptyset \quad \forall a \in A$

2.> $\bigcup_{a \in A} [a] = A$.

- **Prop 14.10:** Let R be an equivalence relation on a set A and let $a, b \in A$. Then $a R b$ iff $[a] = [b]$

Pf:

• Prop. 14.11:

Let R be an equivalence relation on a set A and let $a, x, y \in A$.
If $x, y \in [a]$ then $x R y$.

• Pf: Exercise.

• Proposition 14.12:

Let R be an equivalence relation on a set A and suppose $[a] \cap [b] \neq \emptyset$.
Then $[a] = [b]$.

• Pf:

Cor 14.13: Let R be an equivalence relation on a set A. The equivalence classes of R are nonempty, pairwise disjoint subsets of A whose union is A.