§15 Partitions.

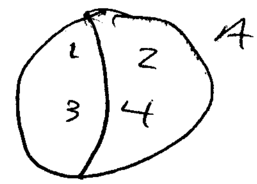
Definition:

Let A be any set. a partition $\mathcal{P}$ of A is a pairwise disjoint set of nonempty subsets of A whose union is A.

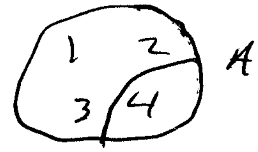
Example:

$$A = \{1, 2, 3, 4\}$$

$$\mathcal{P}_1 = \{\{1,3\}, \{2,4\}\}$$



$$\mathcal{P}_2 = \{\{1,2,3\}, \{4\}\}$$



Observations: If $\mathcal{P}$ is a partition of A then

1) the elements of $\mathcal{P}$ are subsets of A. ($\mathcal{P} \subseteq 2^A$).

2) $\emptyset \notin \mathcal{P}$

3) If $T, S \in \mathcal{P}$ then

$$T \cap S = \begin{cases} \emptyset & \text{if } T \neq S \\ T=S & \text{if } T = S. \end{cases}$$

4) $\bigcup_{S \in \mathcal{P}} S = A.$

- **Note:** We can restate Cor 14.13 as follows.

  Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of A.

- **Definition:** Let $\mathcal{P}$ be a partition of a set A. Define an equivalence relation $\overset{\mathcal{P}}{\equiv}$ on A as follows.

$$\overset{\mathcal{P}}{\equiv} := \{(x,y) : \exists\, S \in \mathcal{P} \text{ s.t. } x, y \in S\}$$

(i.e. $x \overset{\mathcal{P}}{\equiv} y$ iff x and y are in the same part of $\mathcal{P}$.)

Prop 15.3: Let A be a set and let $\mathcal{P}$ be a partition of A. The relation $\overset{\mathcal{P}}{\equiv}$ is an equivalence relation on A.

Pf:

Reflexive:

Symmetric:

Transitive:

Prop 15.4: Let $\mathcal{P}$ be a partition on a set A and let $\overset{\mathcal{P}}{\equiv}$ be as before. The equivalence classes of $\overset{\mathcal{P}}{\equiv}$ are precisely the parts of $\mathcal{P}$.

Pf: Exercise.

How to start?

Let $\mathcal{E} = \{ [a] : a \in A \}$

show that $\mathcal{E} = \mathcal{P}$,

that is show $\mathcal{E} \subset \mathcal{P}$ and $\mathcal{P} \subseteq \mathcal{E}$.

- Example: Let $A = \{1, 2, 3\}$ and let $\mathcal{P}$ be the "has the same size" relation.

What are the equivalence classes of A?

- <u>Example</u>:

  How many rearrangements of

  and

  are there?

  There are

- <u>Example (harder)</u>:

  How many rearrangements of

  all

  are there?

  - <u>Problem</u>:
  - <u>Solution</u>:

  There are ___ ways.

Reexamine:

There are $6 = 3!$ 3 elt. lists from $\{a, l, L\}$ w/ no repeats.

We define a relation $\equiv$ on

$$L = \{ \text{3 elt. lists from } \{a, l, L\} \text{ w/ no repeats} \}$$

by

$$w_1 \equiv w_2$$

if changings all L's to l's will make w_1 and w_2 the same.

- e.g.

$$laL \equiv Lal$$

- **Note:** $\equiv$ is an equiv. relation
  1.) ref. ✓
  2.) sym ✓
  3) trans.

We counted the equiv. classes of $\equiv$!
(i.e. $\equiv$ defines our notion of "sameness").

eg: How many rearrangements of

MISSISSIPPI

are there.

1.) Label as follows

$$\underbrace{M\, I_1\, S_1\, S_2\, I_2\, S_3\, S_4\, I_3\, P_1\, P_2\, I_4}$$

$\#\{$ 11-elt. lists w/no repeats from $\{ \cdot \}\}$

is ________.

Define $\equiv$ on $\mathcal{L}$ by

$w_1 \equiv w_2$

provided that ~~whaen~~ we remove the subscripts from the letters w_1 & w_2 become the same

(eg: $M I_1 S_4 S_3 I_2 S_2 S_1 I_3 P_1 P_2 I_4 = M I_1 S_1 S_2 I_2 S_3 S_4 I_3 P_1 P_2 I_4$)

Q> what is the size of an equiv class?

The number of ways to resubscript.

= ______________ .

So, $\mathcal{L}$ which has size $11!$ is divided into N equiv. classes <u>each of size</u> ______________ .

$$\therefore \quad |\mathcal{L}| = N \cdot (4!\,4!\,2!)$$

$$\Rightarrow N = \frac{11!}{4!\,4!\,2!}$$

$$= \frac{11 \cdot \cancel{10}^{5} \cdot \cancel{9}^{3} \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6}^{3} \cdot 5}{\cancel{(4)}\cancel{(3)}\cancel{(2)}\cancel{(1)}\ \cancel{(2)}\cancel{(1)}}$$

$$= (11)(5)(3)(2)(7)(3)(5)$$

There are $\bullet$ rearrangements.

- Thm:

Let R be an e.r. on a finite set A.
If all the equiv. classes of R have the same size m, then

$$\# \text{ equiv. classes} = \frac{|A|}{m}$$

eg: $A = 2^{\{1,2,3,4\}}$ $\quad (|A| = 2^4 = 16)$

$R = \{ (B,C) : B, C \subseteq \{1,2,3,4\} ; |B| = |C| \}$

equiv. class	size
$[\emptyset] = \{\emptyset\}$	1
$[\{1\}] = \{ \{1\} \{2\} \{3\} \{4\} \}$	4
$[\{1,2\}] = \{ \{1,2\} \{1,3\}, \{1,4\} \{2,3\}, \{2,4\} \{3,4\} \}$	6
$[\{1,2,3\}] = \{ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\} \}$	4
$[\{1,2,3,4\}] = \{ \{1,2,3,4\} \}$	1

Thus, e.c.'s of R have different sizes.