§ 16 Binomial Coefficients

Definition 16.1:

Let \( n, k \in \mathbb{N} \). The symbol \( \binom{n}{k} \) denotes the number of \( k \)-element subsets of an \( n \)-element set. \( \binom{n}{k} \) is read "\( n \) choose \( k \)".

Example:\[ \binom{5}{0} = \quad \cdot \quad \binom{5}{5} = \quad . \]

Fact:\[ \binom{n}{1} = \quad , \quad \binom{n}{n} = \quad . \]

Example:\[ \binom{5}{1} = \quad , \quad \binom{5}{4} = \quad . \]

Fact:\[ \binom{n}{1} = \quad , \quad \binom{n}{n-1} = \quad . \]

Proposition 16.7: Let \( n, k \in \mathbb{N} \) with \( 0 \leq k \leq n \). Then
\[ \binom{n}{k} = \binom{n}{n-k} . \]
Proof: (Combinatorial)

Note: Now we know \( \binom{5}{2} = \binom{5}{3} \) even though we don't know either one at this point.

Example: Evaluate \( \binom{5}{2} \) and \( \binom{5}{3} \).

Proposition 16.5, \( \binom{n}{2} = \binom{n}{n-2} = \)
Proof:

There are \( n-j \) subsets of size 2 whose smallest element is \( j \) for \( 1 \leq j \leq n-1 \). Thus the number of size 2 subsets is

\[
\sum_{j=1}^{n-1} n-j = \sum_{l=1}^{n-1} l
\]

Fact: \[
\sum_{l=1}^{n-1} l = \frac{n(n-1)}{2}
\]

PF: (Gauss's PF)

Example: Compute \( \binom{n}{k} \) for \( 0 \leq k \leq 5 \).

2. Expand \( (x+y)^5 = \)
Theorem 16.8 (Binomial Theorem).

Let \( n \in \mathbb{N} \). Then,

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

Proof:

Example: \((x+y)^2 = \)

Note: \((x+y)^n = \frac{(x+y)(x+y) \cdots (x+y)}{n!} \)
Pascal's Triangle:

- 0th row
- 1st row
- 2nd row

Claim: The \( n \)th row is just \( \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n} \).

Theorem 16.10 (Pascal's Identity):

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

Proof: (Combinatorial)

How many subsets of \( \{1, 2, \ldots, n\} \) have exactly \( k \) elements?

Answer 1: By definition, \( \binom{n}{k} \) is the answer.

Answer 2:

\[
\# \text{k-element subsets} = \# \text{k-element subsets containing } n \\
+ \# \text{k-element subsets not containing } n
\]
Formula for $\binom{n}{k}$:

Let $L = \{ k \text{ elt. lists from } \{1, 2, \ldots, n\} \}$

Then $L = n(n-1)(n-2) \ldots (n-k+1)$

$= \binom{n}{k}$

Define a relation $\equiv$ on $L$ by:

$l_1 \equiv l_2$

provided $l_1$ can be rearranged to get $l_2$

(i.e., $l_1$ & $l_2$ are the same as sets)

$\therefore (1, 2, 3) \equiv (2, 3, 1)$

Note: $\equiv$ is an eq. on $L$

Also the # of $k$-elt. subsets of $\{1, 2, \ldots, n\}$

is the same as the # of equiv. classes of $\equiv$,

Let $S \subseteq L$. Then $[l_0] = \{ l \in L : l \text{ is a rearrangement of } l_0 \}$

$| [l_0] | = k!$

Thus $\binom{n}{k} = \# \text{ of k-element subsets of } \{1, 2, \ldots, n\} = \frac{(n)k}{k!}$. 

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Thm:
\[
\binom{n}{k} = \frac{n!}{k! (n-k)!}
\]

pf:
we saw
\[
\binom{n}{k} = \frac{(n)_k}{k!}
\]

Note: \( (n)_k = \frac{n!}{(n-k)!} \)
\[
\therefore \binom{n}{k} = \frac{n!}{(n-k)! k!}
\]