§16 Binomial Coefficients.

- **Definition 16.1:**
  Let $n, k \in \mathbb{N}$, The symbol $\binom{n}{k}$ denotes the number of k-element subsets of an n-element set.
  ($\binom{n}{k}$ is read "n choose k").

Example: $\binom{5}{0} = \_\_\_\_\_$. $\binom{5}{5} = \_\_\_\_\_$.

Fact: $\binom{n}{0} = \_\_\_\_$, $\binom{n}{n} = \_\_\_\_$.

Example: $\binom{5}{1} = \_\_\_\_\_$, $\binom{5}{4} = \_\_\_\_\_$.

Fact: $\binom{n}{1} = \_\_\_\_$, $\binom{n}{n-1} = \_\_\_\_$.

- **Proposition 16.4:** Let $n, k \in \mathbb{N}$ with $0 \le k \le n$. Then

$$\binom{n}{k} = \binom{n}{n-k},$$

Proof: (Combinatorial)

- Note: Now we know $\binom{5}{2} = \binom{5}{3}$ even though we don't know either one at this point.
- Example: Evaluate $\binom{5}{2}$ and $\binom{5}{3}$.

Proposition 16.5. $\binom{n}{2} = \binom{n}{n-2} =$

- **Proof:**

There are $n-j$ subsets of size 2 whose smallest element is j for $1 \le j \le n-1$.
Thus the # of size 2 subsets is

$$\sum_{j=1}^{n-1} n-j = \sum_{l=1}^{n-1} l$$

$$\boxed{\begin{array}{l} l = n-j \\ j=1 \rightarrow l=n-1 \\ j=n-1 \rightarrow l=1 \end{array}}$$

- **Fact:** $\sum_{l=1}^{n-1} l = \frac{n(n-1)}{2}$

- **Pf.** (Gauss's Pf.)

Example: 1) Compute $\binom{5}{k}$ for $0 \le k \le 5$.

2.) Expant $(x+y)^5 =$

- **Theorem 16.8 (Binomial Theorem).**

Let $n \in \mathbb{N}$. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- **Proof:**

Example: $(x+y)^2 =$

Note: $(x+y)^n = \underbrace{(x+y)}_{1}\underbrace{(x+y)}_{2} \cdots \underbrace{(x+y)}_{n-1}\underbrace{(x+y)}_{n}$.

- **Pascal's Triangle:**

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

– 0^{th} row
– 1^{st} row
– 2^{nd} row

Claim: The n^{th} row is just $\binom{n}{0}\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$.

- **Theorem 16.10 (Pascal's identity).**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- **Proof:** (Combinatorial)

How many subsets of $\{1, 2, \ldots, n\}$ have exactly k elements?

Answer 1: By definition, $\binom{n}{k}$ is the answer.

- Answer 2:

\# k element subsets = \# k element subsets containing n
+ \# k element subsets not containing n.

Formula for $\binom{n}{k}$:

Let $L := \{$ k elt. lists w/no repeats from $\{1,2,\dots,n\}\}$

Then $|L| = n(n-1)(n-2)\cdots(n-k+1)$

$= (n)_k$

Define a relation $\overset{s}{\equiv}$ on L by.

$$l_1 \overset{s}{\equiv} l_2$$

provided l_1 can be rearranged to get l_2

(i.e. l_1 & l_2 are the same as sets)

(~~eg.~~ $(1,2,3) \overset{s}{\equiv} (2,3,1)$)

Note: $\overset{s}{\equiv}$ is an e.r. on L

Also the # of k-elt. subsets of $\{1,2,\dots,n\}$ is the same as the # of equiv. classes of $\overset{s}{\equiv}$.

$\forall\, l_0 \in S$. Then $[l_0] = \{l \in L : l$ is a rearrangement of $l_0\}$

$$|[l_0]| = k!$$

Thus, $\binom{n}{k} = \#$ k element subsets of $\{1,2,\dots,n\} = \dfrac{(n)_k}{k!}$.

Thm:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

pf:

we saw

$$\binom{n}{k} = \frac{(n)_k}{k!}$$

Note: $$(n)_k = \frac{n!}{(n-k)!}$$

$$\therefore \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$