§17 Counting Multisets.

- **Def:** A <u>multiset</u> is an unordered collection of objects in which repetition is allowed.

- **Example:**

$$\langle 1, 1, 2, 3 \rangle$$

$$\langle 1, 2, 3 \rangle = \langle 3, 1, 2 \rangle$$

Def 17.1: Let $n, k \in \mathbb{N}$. The symbol $\left(\!\binom{n}{k}\!\right)$ denotes the number of multisets of size k whose elements are $1, 2, 3, \dots, n-1$ or n (where repetition is allowed).

- **Example:** 1.) Evaluate $\left(\!\binom{n}{0}\!\right)$.

2.) Evaluate $\left(\!\binom{n}{1}\!\right)$.

3.) Evaluate $\left(\!\binom{1}{k}\!\right)$.

Example: Evaluate $\left(\!\!\binom{2}{2}\!\!\right)$.

Example: Evaluate $\left(\!\!\binom{3}{3}\!\!\right)$

$\left(\!\!\binom{n}{k}\!\!\right)$ $\leftarrow k \rightarrow$; $\uparrow n \downarrow$

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462

• **Proposition 17.6:**

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right).$$

• **Example:** Consider multisets of size 3 from $\{1,2,3\}$.

$$\left(\!\!\binom{3}{3}\!\!\right) \begin{cases} \left(\!\!\binom{2}{3}\!\!\right) \left\{ \langle 1,1,1\rangle \quad \langle 1,1,2\rangle \quad \langle 1,2,2\rangle \quad \langle 2,2,2\rangle \right. \\ \left(\!\!\binom{3}{2}\!\!\right) \left\{ \begin{array}{lll} \langle 1,1,\underline{3}\rangle & \langle 1,3,\underline{3}\rangle & \langle 3,3,\underline{3}\rangle \\ \langle 1,2,\underline{3}\rangle & \langle 2,3,\underline{3}\rangle & \langle 2,2,\underline{3}\rangle \end{array} \right. \end{cases}$$

• **Proof:** (combinatorial proof)

How many multisets of size k are there if the elements are $1, 2, \ldots$ or n?

Answer 1: $\left(\!\!\binom{n}{k}\!\!\right)$ by definition.

Answer 2:

\# multisets of size k from $1, 2, \ldots, n$

$=$ \# multisets of size k with no "n"'s

$+$ \# multisets of size k with at least one "n"

$=$

• **Theorem 17.8:** $\forall\, n,k \in \mathbb{N}$, Then
$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

• **Pf:** We will demonstrate a correspondence between multisets of size k whose elements are $1, 2, \ldots$ or n and lists of k $*$'s and $n-1$ $|$'s, and then w/ subsets of $\{1,2,\ldots,n+k-1\}$ of size k.

• **Example:** Representations of $\left(\!\!\binom{3}{3}\!\!\right)$.

Multiset	Stars & Bars	Size 3 Subsets of $\{1,2,3,4,5\}$
$\langle 1,1,1\rangle$	$\underbrace{***}_{3\ 1\text{'s}}\mid\mid$	$\{1,2,3\}$
	no 2's no 3's	
$\langle 1,1,2\rangle$	$**\mid*\mid$	$\{1,2,4\}$
$\langle 1,1,3\rangle$	$**\mid\mid*$	$\{1,2,5\}$
$\langle 1,2,2\rangle$	$*\mid**\mid$	$\{1,3,4\}$
$\langle 1,2,3\rangle$	$*\mid*\mid*$	$\{1,3,5\}$
$\langle 1,3,3\rangle$	$*\mid\mid**$	$\{1,4,5\}$
$\langle 2,2,2\rangle$	$\mid***\mid$	$\{2,3,4\}$
$\langle 2,2,3\rangle$	$\mid**\mid*$	$\{2,3,5\}$
$\langle 2,3,3\rangle$	$\mid*\mid**$	$\{2,4,5\}$
$\langle 3,3,3\rangle$	$\mid\mid***$	$\{3,4,5\}$

Pf: (continued)

Given a multiset S, we produce a pattern of $*$'s & $|$'s as follows.

Write as many $*$'s as there are 1's in S.
Write a $|$
Write as many $*$'s as there are 2's in S.
Write a $|$
⋮
write a $|$
Write as many $*$'s as there are n's in S.

Since this process is invertible, the # of multisets of size k whose elements are 1, 2, ... or n is the same as the number of patterns of k $*$'s and n-1 $|$'s.

To count the number of patterns of k $*$'s and (n-1) $|$'s we simply note that there are $(n-1)+k = n+k-1$ positions to be filled and we need to select k of these in which to place a $*$. Since the order of $*$'s is unimportant and we can't have two symbols in the same position, there are $\binom{n+k-1}{k}$ ways to do this. ∎

§18 Inclusion / Exclusion.

Idea:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$|A \cup B \cup C| =$

$|A \cup B \cup C \cup D| =$