

Suppose we wanted to prove a statement of the form " $A \rightarrow B$ ". If the statement were not true, then there would have to be a counterexample.

So, we suppose there is a counterexample and derive a contradiction. We then conclude that there could be no such counterexample. $\therefore$ the statement " $A \rightarrow B$ " must be true.

Proof Template (Proof by smallest counterexample)
Suppose we want to prove the result $A \rightarrow B$.

- ① Let x be the smallest counterexample to the result you want to prove. It must be clear that there can be such an x .
 - ② Rule out x being the very smallest possibility (basis step). Show there are instances smaller than x satisfying $A \rightarrow B$.
 - ③ Consider an instance of x' of the result that is "just" smaller than x . Use the fact that the result must be true for x' , but the result for x is false to reach a contradiction.
 - ④ Conclude that the result must be true.
- * The result is the thing you want to prove, i.e.,

$$"A \rightarrow B"$$

- Propⁿ 20.1: Every natural number is either odd or even.

Proof:

Cor 20.2: Every integer is either odd or even.

Proof:

- Propⁿ 20.3: Let n be a natural number. The sum of the first n odd natural numbers is n^2 .

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

or $\sum_{k=1}^n (2k-1) = n^2.$

Proof:

Statement 20.62 (Well Ordering Principle) (104)

Every nonempty set of natural numbers contains a smallest element.

-egs:

① $S = \{x \in \mathbb{N}; x \text{ is prime}\}$

smallest element = _____.

② $Y = \{y \in \mathbb{Q}; y > 0\}$

has no smallest element.

Note: Y is not a set of natural numbers.

③ $X = \{x \in \mathbb{N}; x \text{ is both even and odd}\}$

X has no smallest element because

$$X = \emptyset.$$

Proof Template 16.

(Proof by the well-ordering principle)

(105)

To prove a statement about natural numbers:

1.) ~~assume~~ ^{assume} for the sake of contradiction that the statement is false.

2.) Let $X \subseteq \mathbb{N}$ be the set of counter examples.

Since we are assuming the statement is false, $X \neq \emptyset$.

3.) By the W.O. principle X contains a smallest element, say x .

4.) Basis step: Show that the statement is true for 0. Thus $x \neq 0$.

5.) ~~Pick an element $0 \leq y < x$~~

Pick a natural number $0 \leq y < x$.
(maybe $y = x-1$). Then $y \in X$, so the result must hold for y . Argue to a contradiction.

Prop 20.10:

(106)

Let $a \neq 0, 1$. For $n \in \mathbb{N}$,

$$\sum_{k=0}^n a^k = \frac{a^{n+1} - 1}{a - 1}$$

Pf: Let $a \neq 0, 1$.

~~assume~~ ^{assume} for the sake of contradiction, that the statement is false.

Let

$$X = \left\{ n \in \mathbb{N} : \sum_{k=0}^n a^k \neq \frac{a^{n+1} - 1}{a - 1} \right\}$$

We are assuming that the statement is false.

Thus $X \neq \emptyset$.

By the well ordering principle X contains a least element $x \in X$.

• Basis Step: (show $x \neq 0$).

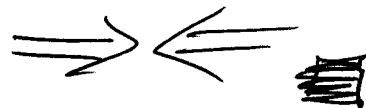
Thus $x > 0$.

Take $y = x - 1$. Then $0 \leq y < x$. Thus $y \notin X$.

So,

$$\sum_{k=0}^y a^k = \frac{a^{y+1} - 1}{a - 1}$$

(i.e., $\sum_{k=0}^{x-1} a^k = \frac{a^x - 1}{a - 1}$.)



Prop 20.11; $\forall n \geq 5, 2^n > n^2$.

- Pf:

n	2^n	n^2
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25
6	64	36
7	128	49
8	256	64

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Assume for the sake of contradiction that the statement is false.

Let $X = \{n \in \mathbb{N} : n \geq 5; 2^n \leq n^2\}$.

By our assumption $X \neq \emptyset$.

By the Well Ordering Principle, X has a smallest element, say x .

(Base step) By our chart, $x > \underline{\hspace{2cm}}$.

Thus $x-1 \geq 5$ and $x-1 \notin X$.

So, $2^{x-1} < (x-1)^2$

Def. 20.12: (Fibonacci Numbers)

$$F_0 = F_1 = 1, \text{ and}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

eg:

n	F_n
0	1
1	1
2	2
3	3
4	
5	
6	
7	

Proposition 20.13: For all $n \in \mathbb{N}$ $F_n \leq 1.7^n$

Pf. Assume for the sake of contradiction that the statement is true.

$$\text{Let } X = \{n \in \mathbb{N}; F_n > 1.7^n\}$$

By our assumption $X \neq \emptyset$.

By well-ordering, there is a smallest element of X , say x .

By our example $x \geq 3$ ($\because 1.7^2 = 2.89 > 2 = F_2$)

Note: $F_x = F_{x-1} + F_{x-2}$