§20: Smallest Counterexample

Suppose we wanted to prove a statement of the form "$A \to B$". If the statement were not true, then there would have to be a counterexample. So, we suppose there is a counterexample and derive a contradiction. We then conclude that there could be no such counterexample. ∴ the statement "$A \to B$" must be true.

Proof Template (Proof by smallest counterexample)
Suppose we want to prove the result $A \to B$.

1. Let x be the smallest counterexample to the result you want to prove. It must be clear that there can be such an x.
2. Rule out x being the very smallest possibility (basis step). Show there are instances smaller than x satisfying $A \to B$
3. Consider an instance of x' of the result that is "just" smaller than x. Use the fact that the result must must be true for x', but the result for x is false to reach a contradiction.
4. Conclude that the result must be true

* The result is the thing you want to prove, i.e., "$A \to B$".

- **Prop^n** 20.1: Every natural number is either odd or even.

Proof:

Cor 20.2: Every integer is either odd or even.

Proof:

- **Prop$^{\underline{n}}$ 20.3:** Let n be a natural number. The sum of the first n odd natural numbers is n^2.

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

or $$\sum_{k=1}^{n} (2k-1) = n^2.$$

Proof:

Statement 20.6: (Well Ordering Principle)

Every nonempty set of natural numbers contains a smallest element.

e.g.:

(1.) $S = \{x \in \mathbb{N} ; \ x \text{ is prime}\}$

smallest element = ______.

(2) $Y = \{y \in \mathbb{Q} : y > 0\}$

has no smallest element.

Note: Y is not a set of natural numbers.

(3.) $X = \{x \in \mathbb{N} : x \text{ is both even and odd.}\}$

X has no smallest element because $X = \emptyset$.

Proof Template 16.

(Proof by the well-ordering principle)

To prove a statement about natural numbers:

1. assume for the sake of contradiction that the statement is false.
2. Let $X \subseteq \mathbb{N}$ be the set of counter examples.
   Since we are assuming the statement is false, $X \neq \emptyset$.
3. By the W.O. principle X contains a smallest element, say x.
4. <u>Basis step:</u> Show that the statement is true for 0. Thus $x \neq 0$.
5. ~~Pick an element~~
   Pick a natural number $0 \leq y < x$. (maybe $y = x-1$). Then $y \notin X$, so the result must hold for y. Argue to a contradiction.

Prop 20.10:

Let $a \neq 0, 1$. For $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$$

Pf: Let $a \neq 0, 1$.

~~Suppose~~ assume for the sake of contradiction, that the statement is false.

Let

$$X = \left\{ n \in \mathbb{N} : \sum_{k=0}^{n} a^k \neq \frac{a^{n+1}-1}{a-1} \right\}$$

We are assuming that the statement is false.

Thus $X \neq \emptyset$.

By the Well Ordering Principle X contains a least element $x \in X$.

• Basis Step: (show $x \neq 0$).

Thus $x > 0$.

Take $y = x-1$. Then $0 \leq y < x$. Thus $y \notin X$.

So,

$$\sum_{k=0}^{y} a^k = \frac{a^{y+1}-1}{a-1}$$

$$\left(\text{i.e., } \sum_{k=0}^{x-1} a^k = \frac{a^{x}-1}{a-1} .\right)$$

$\Rightarrow\Leftarrow$

Prop 20.11: $\forall n \geq 5,\ 2^n > n^2.$

Pf:

n	2^n	n^2	
0	1	0	
1	2	1	
2	4	4	*
3	8	9	*
4	16	16	*
5	32	25	
6	64	36	
7	128	49	
8	256	64	

Assume for the sake of contradiction that the statement is false.

Let $X = \{n \in \mathbb{N} : n \geq 5;\ 2^n \leq n^2\}$.

By our assumption $X \neq \emptyset$.

By the Well Ordering Principle, X has a smallest element, say x.

(Basis step) By our chart, $x >$ ______ .

Thus $x - 1 \geq 5$ and $x - 1 \notin X$.

So, $2^{x-1} < (x-1)^2$

Def. 20.12: (Fibonacci Numbers)

$$F_0 = F_1 = 1, \text{ and}$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

eg:

n	F_n
0	1
1	1
2	2
3	3
4	
5	
6	
7	

Proposition 20.13: For all $n \in \mathbb{N}$ $F_n \leq 1.7^n$

Pf: assume for the sake of contradiction that the statement is true.

Let $X = \{ n \in \mathbb{N} : F_n > 1.7^n \}$

By our assumption $X \neq \emptyset$.

By well-ordering, there is a smallest element of X, say x.

By our example $x \geq 3$ $(\because 1.7^2 = 2.89 > 2 = F_2)$

Note $F_x = F_{x-1} + F_{x-2}$