Section 21 Induction.

- **Theorem 21.2:** (Principle of Mathematical Induction)

  Let A be a set of natural numbers.

  If
  1.) $0 \in A$,
  2.) $\forall k \in \mathbb{N}$, $k \in A \Rightarrow k+1 \in A$

  then $A = \mathbb{N}$.

Pf: (see the book for the proof.)

- **Proof Template:**

  To prove every natural number has some property.

  1.) Let A be the set of natural numbers having the desired property.

  2.) Prove that $0 \in A$. (basis step)

  3.) Prove the statement

  If $k \in A$ then $k+1 \in A$.

  - The hypothesis that $k \in A$ is called the inductive hypothesis.
  - The proof that $k \in A \Rightarrow k+1 \in A$ is called the inductive step.

  4.) Use Thm 21.2 to conclude that $A = \mathbb{N}$.

Proposition 21.3: Let $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} k^2 = \frac{(2n+1)(n+1)n}{6}.$$

Proof: We will proceed by induction on n.

Basis step: When $n=0$, we have

L.H.S. =

R.H.S. =

Thus the result holds for $n=0$.

Inductive hypothesis: (is true for $n=k$)

$$\sum_{l=0}^{k} l^2 = \frac{(2k+1)(k+1)k}{6}$$

Inductive Step: (Use the above equation to show the theorem holds when $n=k+1$.)

Prop 21.4: Let $n \in \mathbb{N}$. Then

$$\sum_{\ell=0}^{n-1} 2^{\ell} = 2^{n} - 1.$$

- Pf:

Prop 21.5: Let $n \geq 1$. Then

$$1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = (n+1)! - 1.$$

Prop 21.7:

Let n [redacted] $\in \mathbb{N}$. Then $3 \mid (4^n - 1)$.

Pf:

Strong
- Induction:

Let A be a set, Suppose that we know the following 2 statements are true.

1.) $N \in A$

2.) $\{N, N+1, N+2, \dots, k\} \subseteq A \Rightarrow k+1 \in A$.

Then $A = \{x \in \mathbb{Z} : x \geq N\}$

(Special case: $(N=0)$:

1.) $0 \in A$

2.) $\{1, 2, 3, \dots, k\} \subseteq A \Rightarrow k+1 \in A$.

Then $A = \{x \in \mathbb{Z} : x \geq 1\} = \mathbb{N}$.)

Proof Template:

To prove the statement: "For all integers $x \geq N$, x satisfies P."

1. Check that $x = N$ satisfies P.] Base case.

2. Suppose that $N, N+1, \ldots, k$ all satisfy P.] $\rightarrow$ Induction Hypothesis

3. Under this assumption, show that $k+1$ also satisfies P.] $\rightarrow$ Inductive step.

4. Letting $A = \{x \in \mathbb{Z} : x \text{ satisfies } P\}$
Then we have verified that
   A. $N \in A$ (1. above)
   B. $\{N, N+1, \ldots, k\} \subseteq A \Rightarrow k+1 \in A$
   (steps 2 & 3 above)

Thus it follows by induction that

$$A = \{x \in \mathbb{Z} : x \geq N\}$$

That is, we have proved that if $x \geq N$ is an integer, then x satisfies P ∎

Prop 21.10:
If a polygon with 4 or more sides is triangulated then at least two of the triangles formed are exterior.

Prop. 21.11: Let $n \in \mathbb{N}$. Then

$$\sum_{j=0}^{n} \binom{n-j}{j} = F_n .$$