§ [redacted] 23 Functions:

Def: a relation f is called a <u>function</u> provided that $(a,b), (a,c) \in f$ only if $b = c$

(i.e. If $(a,c) \in f$ then $(*, c) \notin f$ unless $* = a$.)

- eg.:

$f = \{ (1,3), (2,5), (3,6), (4,7) \}$ is a fc.

$Q = \{ \underline{(1,2)}, (2,4), \underline{(1,3)}, (5,7) \}$ is not a fc.

Notation:
[redacted] $(a,b) \in f$, we write $f(a) = b$.

- eg.: With f as above

$f(1) = 3$, $f(2) =$, $f(3) =$

$f(4) =$

- eg. Express the integer function $f(x) = x^3$ as a set of ordered pairs.

$f = \{ \qquad \}$

- <u>Def:</u> The domain and image of a function f are defined as follows.

$$\mathrm{dom}(f) = \{ a : (a, *) \in f \}$$
$$\mathrm{im}(f) = \{ b : (*, b) \in f \}$$

- <u>Example:</u> Suppose $f = \{ (1,2), (2,4), (3,5), (4,7), (5,10) \}$

Then
$$\mathrm{dom}(f) = \{ \qquad\qquad\qquad\qquad \}$$
$$\mathrm{im}(f) = \{ \qquad\qquad\qquad\qquad \}$$

- <u>Definition:</u> Suppose that f is a function and A and B are sets. Then we write

$$f : A \to B$$

provided that

$$\mathrm{dom}(f) = A \quad \text{and} \quad \mathrm{im}(f) \subseteq B.$$

- <u>Proof Template:</u>

To show $f : A \to B$:

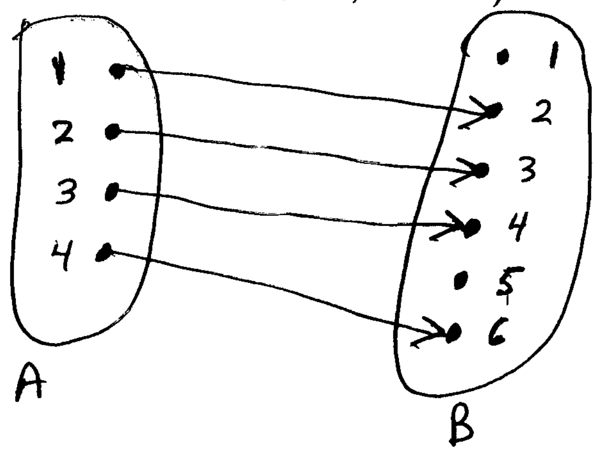
1. Show that f is a function.
2. Show that $\mathrm{dom}(f) = A$ (i.e. $\mathrm{dom}(f) \subset A$ and $A \subset \mathrm{dom}(f)$).
3. Prove $\mathrm{im}(f) \subset B$.

- <u>Example:</u> Let $f = \{ (x, x^2) : x \in \mathbb{Z} \}$

Show that $f : \mathbb{Z} \to \mathbb{N}$.

Pictures of Functions:

- **Example:** $f = \{(1,2),(2,3),(3,4),(4,6)\}$



$$f : A \to B$$

- **Proposition 23.10:** Let A and B be finite sets. The number of functions $f : A \to B$ is given by

$$|B|^{|A|}$$

- **Notation:** We denote the set of functions from $A \to B$ by

$$B^A = \{ f : A \to B \}$$

- **Proof:**

$|B^A| =$ # of $|A|$-element lists whose elements are from B
(i.e. there are $|B|$ choices for each element)

$= |B|^{|A|}$ ∎

- **Example:** List all functions $f: \{1,2,3\} \to \{4,5\}$
  For each function, we will write $(f(1), f(2), f(3))$.

- **Inverse Functions:**
  - **Definition:** A function is one-to-one (1-1) provided that whenever $(x, b), (y, b) \in f$, $x = y$.
    (i.e. $x \neq y \Rightarrow f(x) \neq f(y)$)
  - **Proposition 23.14:** Let f be a function. The inverse relation f^{-1} is a function if and only if f is 1-1.
  - **Proof:** exercise.
  - **Proposition 23.15:** Suppose that f and f^{-1} are functions. Then

$$\operatorname{dom}(f) = \operatorname{im}(f^{-1})$$
$$\operatorname{im}(f) = \operatorname{dom}(f^{-1})$$

  - **Proof:** exercise.
  - **Proof Template:** To show that a function f is 1-1.
    - Direct Method: Suppose that $f(x) = f(y)$. Then argue that $x = y$. (i.e. prove $f(x) = f(y) \Rightarrow x = y$.)
    - Contrapositive: Prove $x \neq y \Rightarrow f(x) \neq f(y)$.
    - Contradiction: Suppose $f(x) = f(y)$ but $x \neq y$ and deduce a contradiction.

- **Example:**
  Let $f(x) = 3x+4$. Prove f is 1-1.
  Proof: Suppose that $f(x) = f(y)$.

- **Definition:** (Onto) Let $f: A \to B$. We say that f is onto provided that $\operatorname{im}(f) = B$.

- **Note:** We already know $\operatorname{im} f \subseteq B$.

- **Proof Template:** Proving $f: A \to B$ is onto.
  Direct Method: Let $b \in B$. Argue that there is $a \in A$ s.t. $f(a) = b$.

  Set Method: Show that $\operatorname{im}(f) = B$.
  (It is only necessary to show that $B \subseteq \operatorname{im}(f)$.)

- **Example:**
  Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x+7$.
  Prove that f is onto.
  Proof: Let $y \in \mathbb{R}$.

eg: Let $f : \mathbb{Q} \to \mathbb{Q}$ be defined by

$$f(x) = 5x + 2$$

Is f 1-1? Is it onto?

- **Thm 23.2(2)** § $f: A \to B$

  $f^{-1}: B \to A$ iff f is 1-1 & onto.

- Pf:

($\Rightarrow$):

($\Leftarrow$):

- Def: A fcn f is called a bijection provided that f is 1-1 and onto

- eg: $f(x) = x+1$ is a bijection from $\mathbb{Z} \to \mathbb{Z}$.

- Pigeon hole principle:

  Let A, B be finite sets and let $f: A \to B$.

  1) $|A| > |B| \Rightarrow f$ is not 1-1.
  2) $|A| < |B| \Rightarrow f$ is not onto.

Prop 23.25 [redacted]: Let A, B be finite sets and let $f: A \to B$. If f is a bijection then $|A| = |B|$

Pf:

Thm 23 [redacted].26: $\$$ A, B are finite sets w/ $|A| = a$ and $|B| = b$. Then

1) $\#\{f: A \to B\} = b^a$

2) $\#\{f: A \to B \mid f \text{ is } 1\text{-}1\} = \begin{cases} (b)_a = \frac{b!}{(b-a)!} & \text{if } a \le b \\ 0 & \text{if } a > b \end{cases}$

3) $\#\{f: A \to B \mid f \text{ is onto}\} = \begin{cases} \sum_{j=0}^{b} (-1)^j \binom{b}{j} (b-j)^a & \text{if } a \ge b \\ 0 & \text{if } a < b \end{cases}$

4) $\#\{f: A \to B \mid f \text{ is a bijection}\} = \begin{cases} a! & \text{if } a = b. \\ 0 & \text{o/w}. \end{cases}$