

§ 23 Functions:

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Def: a relation f is called a function provided that $(a, b), (a, c) \in f$ only if $b = c$

(i.e. If $(a, c) \in f$ then $(*, c) \notin f$ unless $* = a$.)

-eg.:

$f = \{ (1, 3), (2, 5), (3, 6), (4, 7) \}$ is a fc.

$Q = \{ \underline{(1, 2)}, (2, 4), \underline{(1, 3)}, (5, 7) \}$ is not a fc.

Notation:

~~if~~ $(a, b) \in f$, we write $f(a) = b$.

-eg.: with f as above

$$f(1) = 3, f(2) = \quad, f(3) =$$

$$f(4) =$$

-eg. Express the integer function $f(x) = x^3$ as a set of ordered pairs.

$$f = \{ \quad \}$$

• Def: The domain and image of a function are defined as follows.

$$\text{dom}(f) = \{ a : (a, *) \in f \}$$

$$\text{im}(f) = \{ b : (*, b) \in f \}$$

• Example: Suppose $f = \{ (1, 2), (2, 4), (3, 5), (4, 7), (5, 10) \}$

Then

$$\text{dom}(f) = \{ \quad \quad \quad \}$$

$$\text{im}(f) = \{ \quad \quad \quad \}$$

• Definition: Suppose that f is a function and A and B are sets. Then we write

$$f: A \rightarrow B$$

provided that

$$\text{dom}(f) = A \text{ and } \text{im}(f) \subseteq B.$$

• Proof Template:

To show $f: A \rightarrow B$:

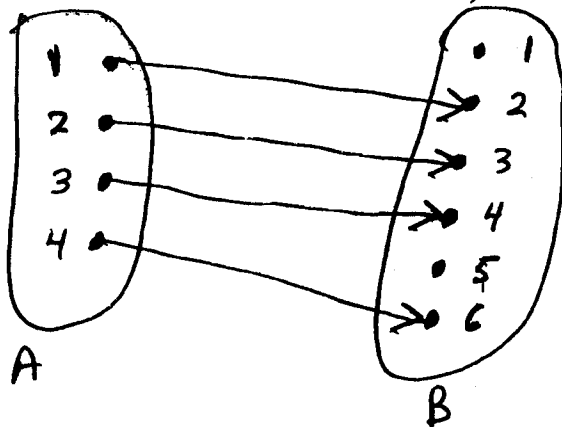
- 1.) show that f is a function.
- 2.) show that $\text{dom}(f) = A$
(i.e., $\text{dom}(f) \subseteq A$ and $A \subseteq \text{dom}(f)$).
- 3.) Prove $\text{im}(f) \subseteq B$.

• Example: Let ~~$f = \{ (x, x^2) : x \in \mathbb{Z} \}$~~ $f = \{ (x, x^2) : x \in \mathbb{Z} \}$
show that $f: \mathbb{Z} \rightarrow \mathbb{N}$.

a Pictures of Functions:

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• Examples $f = \{(1,2), (2,3), (3,4), (4,6)\}$



$$f: A \rightarrow B$$

• Proposition 23.10: Let A and B be finite sets.

The number of functions $f: A \rightarrow B$ is given by

~~$|B|^{|A|}$~~ $|B|^{|A|}$

• Notation: We denote the set of functions from $A \rightarrow B$ by

$$B^A = \{f: A \rightarrow B\}$$

• Proof:

$|B^A| = \#$ of $|A|$ -element lists whose elements are from B

(i.e. there are $|B|$ choices for each element)

$$= |B|^{|A|} \quad \square$$

• Example: List all functions $f: \{1, 2, 3\} \rightarrow \{4, 5\}$
For each function, we will write $(f(1), f(2), f(3))$. (126)

• Inverse Functions:

• Definition: A function is one-to-one (1-1) provided that whenever $(x, b), (y, b) \in f$, $x = y$.
(i.e. $x \neq y \Rightarrow f(x) \neq f(y)$)

• Proposition 23.14: Let f be a function. The inverse relation f^{-1} is a function if and only if f is 1-1.

• Proof: exercise.

• Proposition 23.15: Suppose that f and f^{-1} are functions. Then

$$\begin{aligned}\text{dom}(f) &= \text{im}(f^{-1}) \\ \text{im}(f) &= \text{dom}(f^{-1})\end{aligned}$$

• Proof: exercise.

• Proof Templates: To show that a function f is 1-1.

Direct Method: Suppose that $f(x) = f(y)$. Then argue that $x = y$. (i.e. prove $f(x) = f(y) \Rightarrow x = y$)

Contrapositive: Prove $x \neq y \Rightarrow f(x) \neq f(y)$.

Contradiction: Suppose $f(x) = f(y)$ but $x \neq y$ and deduce a contradiction.

Example:

Let $f(x) = 3x + 4$. Prove f is 1-1.

Proof: Suppose that $f(x) = f(y)$.

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Definition: (onto) Let $f: A \rightarrow B$. We say that f is onto provided that $\text{im}(f) = B$.

Note: We already know $\text{im}(f) \subseteq B$.

Proof Template: Proving $f: A \rightarrow B$ is onto.

Direct Method: Let $b \in B$. Argue that there is $a \in A$ s.t. $f(a) = b$.

Set Method: Show that $\text{im}(f) = B$.

(It is only necessary to show that $B \subseteq \text{im}(f)$.)

Example: ~~Let $f: \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by $f(x) = 3x + 7$.~~

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(x) = 3x + 7$.

Prove that f is onto.

Proof: Let $y \in \mathbb{Z}$.

eg: Let $f: \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by

$$f(x) = 5x + 2$$

Is f 1-1? Is it onto?

Thm 23.2 (i) $f: A \rightarrow B$

$f^{-1}: B \rightarrow A$ iff f is 1-1 & onto.

-pf:
 $(\Rightarrow)$

$(\Leftarrow)$

-Def: A fcn f is called a bijection provided that f is 1-1 and onto

-eg: $f(x) = x+1$ is a bijection from $\mathbb{Z} \rightarrow \mathbb{Z}$.

~~Pigeon hole principle~~
• Pigeon hole principle:

Let A, B be finite sets and let $f: A \rightarrow B$.

1) $|A| > |B| \Rightarrow f$ is not 1-1,

2) $|A| < |B| \Rightarrow f$ is not onto,

Prop ^{23,25} ~~23,25~~: Let A, B be finite sets and
 let $f: A \rightarrow B$. If f is a bijection
 then $|A| = |B|$

Pf:

Thm ²³ ~~23~~, 26: $\nexists$ A, B are finite sets w/ $|A| = a$ and
 $|B| = b$, Then

$$1.) \# \{ f: A \rightarrow B \} = b^a$$

$$2.) \# \{ f: A \rightarrow B \mid f \text{ is 1-1} \} = \begin{cases} (b)_a = \frac{b!}{(b-a)!} & \text{if } a \leq b \\ 0 & \text{if } a > b \end{cases}$$

$$3.) \# \{ f: A \rightarrow B \mid f \text{ is onto} \} = \begin{cases} \sum_{j=0}^b (-1)^j \binom{b}{j} (b-j)^a & \text{if } a \geq b \\ 0 & \text{if } a < b \end{cases}$$

$$4.) \# \{ f: A \rightarrow B \mid f \text{ is a bijection} \} = \begin{cases} a! & \text{if } a = b, \\ 0 & \text{o/w.} \end{cases}$$