§ 24 The Pigeon Hole Principle.

Prop 24.1:
Let $n \in \mathbb{N}$. Then there exists integers a, b with $a \neq b$ and $10 \mid n^a - n^b$

Pf:
Consider

$$n^1, n^2, n^3, n^4, \ldots, n^{11}$$

The ones digit of each is contained in $\{0, 1, 2, \ldots, 9\}$.

Consider a map $f: \mathbb{N} \to \{0, 1, 2, \ldots, 9\}$ defined by $f(x) =$ 1's digit of x.

Prop 24.2:
Given 5 distinct lattice points in the plane (i.e. 5 points in $\mathbb{R}^2$ with integer coordinates), at least one of the line segments determined by these points has a lattice point as its midpoint.

Note: The midpoint of the line segment from (a,b) to (c,d) is given by $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$

Pf: Consider the map
$$f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \{0,1\} \times \{0,1\}$$
given by
$$f((a,b)) = \begin{cases} (0,0), & \text{if } a \text{ and } b \text{ are even} \\ (0,1), & \text{if } a \text{ is even \& } b \text{ is odd} \\ (1,0) & \text{if } a \text{ is odd \& } b \text{ is even} \\ (1,1) & \text{if } a \text{ \& } b \text{ are odd.} \end{cases}$$

Thm 24.4: (Cantor).

Let A be a set. If $f: A \to 2^A$ then f is not onto.

Note: If A is finite then $|2^A| = 2^{|A|} > |A|$. Thus the result follows.

Pf: Let A be a set. We will find $B \in 2^A$ (i.e. $B \subseteq A$) s.t. $B \notin im(f)$.

Note: $f: A \to 2^A$ means that $\forall x \in A,\ f(x) \subseteq A$.

Put

$$B = \{x \in A : x \notin f(x)\}.$$

Claim: There is no $a \in A$ such that $f(a) = B$.

Pf: Two cases $a \in B$ or $a \notin B$.