MTHSC 206 Section 12.3 –Dot Products and Projections

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Definition

Suppose that $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Then we define the dot product $a \cdot b$ of a and b by

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example

$(1, 2, -1) \cdot (2, 3, -2) = (1)(2) + (2)(3) + (-1)(-2) = 2 + 6 + 2 = 10$

Properties of the Dot Product

Suppose that $a, b, c \in \mathbb{R}^3$ and $d \in \mathbb{R}$.

1. $a \cdot a = |a|^2$.
2. $a \cdot b = b \cdot a$.
3. $a \cdot (b + c) = a \cdot b + a \cdot c$.
4. $(da) \cdot b = d(a \cdot b) = a \cdot (db)$.
5. $0 \cdot a = 0$.

THEOREM

Suppose that a and b are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then,

$$a \cdot b = |a||b|\cos(\theta),$$

where θ is the angle between a and b.

COROLLARY

Suppose that a, b and θ are as above. Then

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}.$$

EXAMPLE

The angle between $(1, 1, 1)$ and $(1, 1, 0)$ is $\theta = \arccos(\frac{2}{\sqrt{6}})$.

Fact

Two vectors are perpendicular or orthogonal if and only if $a \cdot b = 0$.

Example

i, j and k are pairwise orthogonal.

Direction Angles

Suppose that $a \in \mathbb{R}^3$ and that α, β and γ are the angles between a and the x-, y- and z-axes (respectively). Then,

$$\cos(\alpha) = \frac{a_1}{|a|}, \quad \cos(\beta) = \frac{a_2}{|a|} \quad \text{and} \quad \cos(\gamma) = \frac{a_3}{|a|}.$$

Note

Suppose that $a \in \mathbb{R}^3$ is a vector and that α, β and γ are its position angles. Then,

1. $\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = 1$.
2. $a = (|a|\cos(\alpha), |a|\cos(\beta), |a|\cos(\gamma)) = |a|(\cos(\alpha), \cos(\beta), \cos(\gamma))$.
3. $\frac{1}{|a|}a = (\cos(\alpha), \cos(\beta), \cos(\gamma))$.

Definition

Suppose that $a = \vec{PQ}$ and $b = \vec{PR}$ $(Q \neq R)$. Let $\mathcal{L}$ be a line which is perpendicular to the line containing a and passes through R. Let S be the intersection point of $\mathcal{L}$ with the line containing a. Then the vector $\vec{PS}$ is the vector projection of b onto a and is denoted by $\mathrm{Proj}_a(b)$.

Definition

The scalar projection of b onto a (or component of b along a) is defined to be the signed magnitude of $\mathrm{Proj}_a(b)$ and is denoted by $\mathrm{comp}_a(b)$.

Note

Suppose that a and b are vectors and that θ is the angle between them.

1. $\text{comp}_a(b) = |b|\cos(\theta)$.
2. $a \cdot b = |a||b|\cos(\theta)$.
3. $\text{comp}_a(b) = \frac{a \cdot b}{|a|}$.
4. $\text{Proj}_a(b) = \text{comp}_a(b)\frac{a}{|a|} = \frac{a \cdot b}{|a|^2}a = \left(\frac{a \cdot b}{a \cdot a}\right)a$.

Example

Let $a = (1, 1, 0)$ and $b = (1, 1, 1)$. Compute the vector and scalar projections of b onto a.

Suppose that a constant force vector F acts on an object moving it from P to Q.
Then the work done is component of F acting in the direction of $D = \vec{PQ}$ multiplied by the distance the object has moved, namely $|PQ|$.
So, if θ is the angle between the force vector F and the displacement vector D, then

$$W = (|F|\cos(\theta))|D| = |F||D|\cos(\theta) = F \cdot D.$$