MTHSC 206 Section 12.4 –Cross Products

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Definition

Suppose that $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_2)$ are vectors. We define the cross products $a \times b$ of a and b as

$$\begin{aligned} a \times b &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k. \end{aligned}$$

Note

This definition will only work for 3 dimensional vectors.

Exercise

1. Let $a = (1, 3, 2)$ and $b = (2, -1, 1)$. Compute $a \times b$.
2. Compute $(a \times b) \cdot a$ and $(a \times b) \cdot b$.

Theorem

Suppose that $a, b \in \mathbb{R}^3$. Then, the vector $a \times b$ is orthogonal to a and to b.

Proof.

$$
\begin{aligned}
(a \times b) \cdot a
&= ((a_2 b_3 - a_3 b_2), (a_3 b_1 - a_1 b_3), (a_1 b_2 - a_2 b_1)) \cdot (a_1, a_2, a_3) \\
&= (a_2 b_3 - a_3 b_2) a_1 + (a_3 b_1 - a_1 b_3) a_2 + (a_1 b_2 - a_2 b_1) a_3 = 0.
\end{aligned}
$$

You can check orthogonality to b. $\square$

Theorem

Suppose that $a, b \in \mathbb{R}^3$ *and that* θ *is the angle between them. Then* $|a \times b| = |a||b|\sin(\theta)$.

Sketch of proof...

$$
\begin{aligned}
|a \times b|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
\ldots &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= |a|^2|b|^2 - (a \cdot b)^2 \\
&= |a|^2|b|^2 - (|a||b|\cos(\theta))^2 \\
&= |a|^2|b|^2(1 - \cos^2(\theta)).
\end{aligned}
$$

$\square$

Corollary

Two nonzero vectors $a, b \in \mathbb{R}^3$ are parallel if and only if $a \times b = 0$.

Fact

The area of the parallelogram determined by $a, b \in \mathbb{R}^3$ is given by $|a \times b|$.

Exercise

Find the area of the triangle with vertices $P = (1, 1, 1)$, $Q = (1, 2, 1)$ and $R = (2, 2, 3)$.

Note

$$i \times j = k \qquad j \times k = i, \qquad k \times i = j.$$
$$j \times i = -k \qquad k \times j = -i, \qquad i \times k = -j.$$

Properties of $\times$

Suppose that $a, b, c \in \mathbb{R}^3$ and $d \in \mathbb{R}$. Then,

1. $a \times b = -b \times a$.
2. $(da) \times b = d(a \times b) = a \times (db)$.
3. $a \times (b + c) = a \times b + a \times c$.
4. $(a + b) \times c = a \times c + b \times c$.
5. $a \cdot (b \times c) = (a \times b) \cdot c$.
6. $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$. In particular $\times$ is not associative.

Theorem

The volume of the parallelepiped determined by $a, b, c \in \mathbb{R}^3$ *is given by* $V = |a \cdot (b \times c)|$.

Proof.

$$
\begin{aligned}
V &= (\text{Area of parallelogram determined by } b \text{ and } c) * (\text{height}) \\
&= (\text{height}) * |b \times c| \\
&= |\text{Proj}_{b\times c}(a)| * |b \times c| \\
&= \left| \frac{(b \times c) \cdot a}{|b \times c|^2} (b \times c) \right| * |b \times c| \\
&= \frac{|(b \times c) \cdot a|}{|b \times c|^2} |b \times c| * |b \times c|
\end{aligned}
$$

□

Exercise

Use the vector triple product to show that the vectors $a = (1, 1, 1)$, $b = (1, 2, 1)$ and $c = (2, 1, 2)$ are coplanar.

Suppose that a force vector F acts on one end a rigid body which is fixed at its other end and is represented by the position vector r. We define the torque vector τ as

$$\tau = r \times F.$$

The torque vector indicates the direction of rotation (using the right-hand rule) and measures the tendency of the object to rotate about the origin.
Note that

$$|\tau| = |r \times F| = |r||F| \sin(\theta),$$

where θ is the angle between F and r.