MTHSC 206 Section 12.5 –Equations of Lines and Planes

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Definition

A line in $\mathbb{R}^3$ can be described by a point and a direction vector. Given the point r_0 and the direction vector v. Any point r on the line through r_0 and parallel to v satisfies $r = r_0 + tv$ for some $t \in \mathbb{R}$.

Definition

If we let $r_0 = (x_0, y_0, z_0)$ and $v = (a, b, c)$, then the line described above could also be described by the parametric equations

$$x = x_0 + at, \quad y = y_0 + bt \quad \text{and} \quad z = z_0 + tc.$$

Definition

If the coordinates of the direction vector are all non zero, then we have a third description of the line given by solving each of the parametric equations for t and equating, namely

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Exercise

Give vector, parametric and symmetric equations for the line passing though the points $A = (1, 1, 1)$ and $B = (0, -1, 2)$. At what point does this line intersect the xy-plane?

Definition

A plane in $\mathbb{R}^3$ can be described by a point r_0 in the plane along with a normal vector n which is orthogonal to all vectors in the plane. Given this information, we see that r is in the plane if and only if $r - r_0$ is orthogonal to n, that is

$$\begin{aligned} n \cdot (r - r_0) &= 0, \quad \text{or} \\ n \cdot r &= n \cdot r_0. \end{aligned}$$

Either of these equations is called the vector equation of the plane.

Note

Letting $n = (a, b, c)$, $r = (x, y, z)$ and $r_0 = (x_0, y_0, z_0)$, we obtain the scalar equation of the plane through the point $r_0 = (x_0, y_0, z_0)$ with normal vector $n = (a, b, c)$, namely

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Note

The last equation can be rewritten in the form

$$ax + by + cz + d = 0.$$

This is called a linear equation of the plane.

Exercise

Find an equation for the plane containing the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Fact

1. *Two planes are parallel if and only if their normal vectors are parallel.*
2. *The angle between two intersecting planes is equal to the angle between their normal vectors.*

Exercise

Find the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 0$. Describe their intersection.

Fact

The distance from the point $P = (x_1, y_1, z_1)$ *to the plane* $ax + by + cz + d = 0$ *is given by*

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Proof.

Note that a normal vector to the plane is $n = (a, b, c)$.
Let P_0 be any point in the plane and let $b = \overrightarrow{P_0P} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$.
Then, $D = |\mathrm{comp}_n(b)| = \frac{|n \cdot b|}{|n|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$.
Now note that $-ax_0 - by_0 - cz_0 = d$, since P_0 is in the plane. □