

# MTHSC 206 SECTION 13.2 – DERIVATIVES AND INTEGRALS OF VECTOR FUNCTIONS

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## DEFINITION

Suppose that  $r(t)$  is a vector function. We define its derivative by

$$\frac{dr}{dt} = r'(t) = \lim_{h \rightarrow 0} \left[ \frac{r(t+h) - r(t)}{h} \right]$$

## NOTE

When  $r'(t_0)$  exists and is nonzero, we refer to it as the tangent vector to the space curve defined by  $r(t)$  at the point  $P = r(t_0)$ . In this case, the line tangent to the curve at  $P = r(t_0)$  is given by  $P + r'(t_0)t$ .

## DEFINITION

The unit tangent vector to the curve is defined by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

## THEOREM

Suppose that  $r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$ , where  $f, g$  and  $h$  are differentiable functions. Then,

$$r'(t) = (f'(t), g'(t), h'(t)) = f'(t)i + g'(t)j + h'(t)k.$$

## EXAMPLE

Find the derivative of  $r(t) = (\cos(t), \sin(t), 2t)$ . Find the unit tangent vector at the point where  $t = \pi$ .

## THEOREM (PROPERTIES OF DERIVATIVES)

Suppose that  $u, v$  are differentiable vector functions,  $c \in \mathbb{R}$  and  $f(t)$  is a real valued function. Then,

$$\textcircled{1} \quad \frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t).$$

$$\textcircled{2} \quad \frac{d}{dt}[cu(t)] = cu'(t).$$

$$\textcircled{3} \quad \frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t).$$

$$\textcircled{4} \quad \frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t).$$

$$\textcircled{5} \quad \frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t).$$

$$\textcircled{6} \quad \frac{d}{dt}[u(f(t))] = f'(t)u'(f(t)).$$

### EXAMPLE

Show that if  $|r(t)| = c$  is constant, then  $r'(t)$  is orthogonal to  $r(t)$  for all  $t$ .

### EXAMPLE

What is the derivative of  $r(t) = (\cos(\sin(t)), \sin(\sin(t)), e^{\sin(t)})$ ?

## DEFINITION

We define the definite integral of a continuous vector function  $r(t) = (f(t), g(t), h(t))$  on an interval  $[a, b]$  as

$$\begin{aligned} \int_a^b r(t) dt &= \lim_{n \rightarrow \infty} \sum_{i=1}^n r(t_i^*) \Delta t \\ &= \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_i^*) \Delta t, \lim_{n \rightarrow \infty} \sum_{i=1}^n g(t_i^*) \Delta t, \lim_{n \rightarrow \infty} \sum_{i=1}^n h(t_i^*) \Delta t \right) \end{aligned}$$

## FACT

If  $r(t) = (f(t), g(t), h(t))$ , then

$$\int_a^b r(t) dt = \left( \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right).$$

## THEOREM (FUNDAMENTAL THEOREM OF CALCULUS)

Suppose that  $R(t)$  and  $r(t)$  are continuous vector valued functions with  $R'(t) = r(t)$ . Then,

$$\int_a^b r(t)dt = [R(t)]\Big|_a^b = R(b) - R(a).$$

## NOTE

If  $R$  and  $r$  are as above, then we will use the notation  $R(t) = \int r(t)dt$  for the indefinite integral of  $r(t)$ .

## EXAMPLE

Compute  $\int_0^{\pi/2} (\cos(t), \sin(t), 2t)dt$ .