MTHSC 206 Section 13.2 – Derivatives and Integrals of Vector Functions

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Definition

Suppose that $r(t)$ is a vector function. We define its derivative by

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The unit tangent vector to the curve is defined by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Theorem

Suppose that $r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$, *where* f, g *and* h *are differentiable functions. Then,*

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Example

Find the derivative of $r(t) = (\cos(t), \sin(t), 2t)$. Find the unit tangent vector at the point where $t = \pi$.

Theorem (Properties of Derivatives)

Suppose that u, v are differentiable vector functions, $c \in \mathbb{R}$ and $f(t)$ is a real valued function. Then,

1. $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$.
2. $\frac{d}{dt}[cu(t)] = cu'(t)$.
3. $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$.
4. $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$.
5. $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$.
6. $\frac{d}{dt}[u(f(t))] = f'(t)u'(f(t))$.

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What is the derivative of $r(t) = (\cos(\sin(t)), \sin(\sin(t)), e^{\sin(t)})$?

DEFINITION

We define the definite integral of a continuous vector function $r(t) = (f(t), g(t), h(t))$ on an interval $[a, b]$ as

$$\int_a^b \quad r(t)\text{dt} = \lim_{n\to\infty} \sum_{i=1}^{n} r(t_i*)\Delta t$$

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FACT

If $r(t) = (f(t), g(t), h(t)$, *then*

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right).$$

Theorem (Fundamental Theorem of Calculus)

Suppose that $R(t)$ *and* $r(t)$ *are continuous vector valued functions with* $R'(t) = r(t)$. *Then,*

$$\int_a^b r(t)dt = [R(t)]|_a^b = R(b) - R(a).$$

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If R and r are as above, then we will use the notation $R(t) = \int r(t)\mathrm{d}t$ for the indefinite integral of $r(t)$.

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Example

Compute $\int_0^{\pi/2} \langle \cos(t), \sin(t), 2t \rangle \text{dt}$.