MTHSC 206 Section 13.3 – Arc Length and Curvature

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Fact

Suppose that $r(t) = (x(t), y(t), z(t))$*. Then the arc length of the segment of the curve defined by* $r(t)$ *where* $a \leq t \leq b$ *is given by*

$$L = \int_a^b |r'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

Example

Compute the length of the arc defined by
$r(t) = (\sin(\sin(t)), \cos(\sin(t)), \cos(t))$ as t varies from 0 to 2π.

Example

Note that the curve defined by $r(t) = (2t, t^2, \frac{1}{3}t^3)$ where $1 \leq t \leq 100$ could also be described by $q(t) = (2e^u, e^{2u}, \frac{1}{3}e^{3u})$, where $0 \leq u \leq \ln 100$.
The relationship between the parameters t and u is the invertible function $t = e^u$ which has inverse $u = \ln t$.

Fact

Our definition of arc length does not depend on the parametrization of the curve. It only depends on the beginning and ending points of the arc.

Example

Compute the arc length of the arc along the curve of the above example from $(2e, e^2, \frac{1}{3}e^3)$ to $(2e^2, e^4, \frac{1}{3}e^6)$.

Definition

Suppose that a curve C is parametrized by the vector function $r(t)$ as $a \leq t \leq b$ and that C is traversed exactly once as t goes from a to b. Then we define the arc length function for C as follows.

$$s(t) = \int_a^t |r'(u)| \mathrm{du}.$$

Note

By the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}s}{\mathrm{dt}} = |r'(t)|.$$

Since the arc length function s is independent of choice of coordinates, it is often desirable to write t in terms of s and then write r as a function of s, namely $r(t(s))$.

Example

Let $r(t) = (\cos(t), \sin(t), t)$. Write r as a function of its arc length.

Definition

A parametrization $r(t)$ is called smooth on an interval I if $r'(t)$ is continuous and nonzero on I.
A curve C is called smooth if it has a smooth parametrization.

Recall

If C is a smooth curve parametrized by $r(t)$, then $T(t) = \frac{r'(t)}{|r'(t)|}$ is its unit tangent vector at the point $r(t)$.
This vector indicates the direction of the curve.

Definition

We define the curvature of a curve by

$$\kappa = \left| \frac{dT}{ds} \right| .$$

Fact

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}.$$

Proof.

Note that T can be written as a function of s.
Then the chain rule give, $T'(t) = T'(s(t))s'(t)$.
That is, $\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} \cdot |r'(t)|$. $\square$

Example

Compute the curvature of the circle or radius a.

Theorem

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

Proof.

Recall that $T = \frac{r'}{|r'|}$.
So, $r' = T|r'| = Ts'$.
$\Rightarrow r'' = T's' + Ts''$.
Thus, $r' \times r'' = (s'T) \times (T's' + Ts'') =$
$(s')^2(T \times T') + (s's'')(T \times T) = (s')^2(T \times T')$.
Recall that $|T| = 1$ which implies that $T \perp T'$.
So, $|r' \times r''| = (s')^2|T \times T'| = (s')^2|T||T'| = (s')^2|T'|$.
Thus $|T'| = \frac{|r' \times r''|}{(s')^2} = \frac{|r' \times r''|}{|r'|^2}$. Therefore, $\kappa(t) = \frac{|T'|}{|r'|} = \frac{|r' \times r''|}{|r'|^3}$. $\square$

Example

Compute the curvature of the curve parametrized by $r(t) = (t, t^2, t^3)$.

Note (2D case)

Suppose that we have a plane curve given by $y = f(x)$.
We can parametrize this curve in $\mathbb{R}^3$ by $r(x) = (x, f(x), 0)$.
So, we have $r'(x) = (1, f'(x), 0)$ and $r''(x) = (0, f''(x), 0)$.
Thus, $r' \times r'' = (i + f'(x)j) \times f''(x)j =$
$f''(x)(i \times j) + f'(x)f''(x)(j \times j) = f''(x)k = (0, 0, f''(x))$.
Therefore, we have $\kappa(x) = \frac{\sqrt{(f''(x))^2}}{\sqrt{1+(f'(x))^2}^3} = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$.

Definition

Given a curve C parametrized by $r(t)$, we define the principal unit normal vector of C at the point $r(t)$ as

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$

Fact

$$N(t) \perp T(t).$$

Definition

We define the binormal vector of C at $r(t)$ as

$$B(t) = T(t) \times N(t).$$

Example

Consider the curve C parametrized by $r(t) = (\cos(t), \sin(t), 3t)$. Compute the unit tangent, the unit normal and the binormal vectors at $r(\pi) = (-1, 0, 3\pi)$.

Definition

The plane determined by $N(t)$ and $B(t)$ is called the normal plane of C at $P = r(t)$.
The plane determined by $T(t)$ and $N(t)$ is called the osculating plane of C at $P = r(t)$ or tangent plane of C at P.

Note

The normal plane at $r(t)$ has normal vector $T(t)$.
The tangent plane at $r(t)$ has normal vector $B(t)$.

Example

Find equations of the normal and osculating planes of the curve C parametrized by $r(t) = (\cos(t), \sin(t), 3t)$ at the point $(-1, 0, 3\pi)$.

Definition

The circle that lies in the osculating plane of C at P, has the same tangent as C and lies on the concave side of C (-i.e. in the direction N points) and has radius $\rho = \frac{1}{\kappa(t)}$ is called <u>the osculating circle</u> of C at P.

Note

Let C be a curve and S its osculating circle. Then S has the same curvature as C at P. That is, S is the circle that best indicates the behavior of the curve C near P.

Example

Find and graph the osculating circle of the curve C parametrized by $r(t) = (\cos(t), \sin(t), 3t)$ at the point $(-1, 0, 3\pi)$.