# MTHSC 206 Section 13.3 – Arc Length and Curvature

Kevin James

## Fact

*Suppose that* $r(t) = (x(t), y(t), z(t))$. *Then the arc length of the segment of the curve defined by* $r(t)$ *where* $a \leq t \leq b$ *is given by*

$$L = \int_a^b |r'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

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## Example

Compute the length of the arc defined by
$r(t) = (\sin(\sin(t)), \cos(\sin(t)), \cos(t))$ as $t$ varies from 0 to $2\pi$.

### Example

Note that the curve defined by $r(t) = (2t, t^2, \frac{1}{3}t^3)$ where $1 \leq t \leq 100$ could also be described by $q(t) = (2e^u, e^{2u}, \frac{1}{3}e^{3u})$, where $0 \leq u \leq \ln 100$.

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## Example

Compute the arc length of the arc along the curve of the above example from $(2e, e^2, \frac{1}{3}e^3)$ to $(2e^2, e^4, \frac{1}{3}e^6)$.

## Definition

Suppose that a curve $C$ is parametrized by the vector function $r(t)$ as $a \leq t \leq b$ and that $C$ is traversed exactly once as $t$ goes from $a$ to $b$. Then we define the arc length function for $C$ as follows.

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$$s(t) = \int_a^t |r'(u)| \mathrm{d}u.$$

**Note**

By the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |r'(t)|.$$

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**EXAMPLE**

Let $r(t) = (\cos(t), \sin(t), t)$. Write $r$ as a function of its arc length.

### Definition

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This vector indicates the direction of the curve.

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We define the curvature of a curve by

$$\kappa = \left|\frac{dT}{ds}\right|.$$

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## Example

Compute the curvature of the circle or radius $a$.

## Theorem

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

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Therefore, we have $\kappa(x) = \frac{\sqrt{(f''(x))^2}}{\sqrt{1+(f'(x))^2}^3} = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$.

**Definition**

Given a curve $C$ parametrized by $r(t)$, we define the principal unit normal vector of $C$ at the point $r(t)$ as

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$

# The Normal and Binormal vectors

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### Definition

We define the binormal vector of $C$ at $r(t)$ as

$$B(t) = T(t) \times N(t).$$

## Example

Consider the curve $C$ parametrized by $r(t) = (\cos(t), \sin(t), 3t)$. Compute the unit tangent, the unit normal and the binormal vectors at $r(\pi) = (-1, 0, 3\pi)$.

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## Definition

The plane determined by $N(t)$ and $B(t)$ is called the normal plane of $C$ at $P = r(t)$.
The plane determined by $T(t)$ and $N(t)$ is called the osculating plane of $C$ at $P = r(t)$ or tangent plane of $C$ at $P$.

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## Note

The normal plane at $r(t)$ has normal vector $T(t)$.
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The normal plane at $r(t)$ has normal vector $T(t)$.
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### Example

Find equations of the normal and osculating planes of the curve $C$ parametrized by $r(t) = (\cos(t), \sin(t), 3t)$ at the point $(-1, 0, 3\pi)$.

## Definition

The circle that lies in the osculating plane of $C$ at $P$, has the same tangent as $C$ and lies on the concave side of $C$ (-i.e. in the direction $N$ points) and has radius $\rho = \frac{1}{\kappa(t)}$ is called the osculating circle of $C$ at $P$.

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## Note

Let $C$ be a curve and $S$ its osculating circle. Then $S$ has the same curvature as $C$ at $P$. That is, $S$ is the circle that best indicates the behavior of the curve $C$ near $P$.

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## Example

Find and graph the osculating circle of the curve $C$ parametrized by $r(t) = (\cos(t), \sin(t), 3t)$ at the point $(-1, 0, 3\pi)$.