MTHSC 206 Section 13.4 – Velocity and Acceleration

Kevin James

Definition

Suppose that an object's position at time t is given by the vector function $r(t)$. Then,

1. The velocity vector of the object at time t is given by $v(t) = r'(t)$.
2. The speed of the object is given by $|v(t)| = |r'(t)| = \frac{ds}{dt}$.
3. The acceleration vector of the object at time t is given by $a(t) = v'(t) = r''(t)$.

Newton's second law of motion

$$F(t) = ma(t).$$

Example

Suppose that a projectile is to be fired into the air at an angle of α from the ground with an initial velocity vector of magnitude v_0. What is the position function for the projectile? What angle will maximize the distance the projectile will travel before returning to the ground. You may assume that the only external force acting on the projectile is gravity. (Acceleration due to gravity is 9.8 m/s^2.)

Letting $\mathfrak{s}(t) = |v(t)|$ denote the speed of the particle. Then we have

$$\begin{aligned} T(t) &= \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{\mathfrak{s}(t)} \Rightarrow v = T\mathfrak{s} \\ \Rightarrow a &= T'\mathfrak{s} + T\mathfrak{s}' \end{aligned}$$

Recall that

$$\begin{aligned} \kappa = \frac{|T'|}{|r'|} = \frac{|T'|}{\mathfrak{s}} \quad &\Rightarrow \quad |T'| = \kappa\mathfrak{s}. \quad \text{and} \\ N = \frac{T'}{|T'|} \quad &\Rightarrow \quad T' = N|T'| = N\kappa\mathfrak{s}. \end{aligned}$$

So, $a = \mathfrak{s}'T + \kappa\mathfrak{s}^2 N$.

So, we have $a = a_T T + a_N N$, wherer $a_T = \mathfrak{s}'$ and $a_N = \kappa \mathfrak{s}^2$.
Note that $v \cdot a = \mathfrak{s} T \cdot (\mathfrak{s}' T + \kappa \mathfrak{s}^2 N) = \mathfrak{s}\mathfrak{s}'$.
Thus, $a_T = \frac{v \cdot a}{\mathfrak{s}} = \frac{r' \cdot r''}{|r'|}$.
By our theorem on curvature, we have
$a_N = \kappa \mathfrak{s}^2 = \frac{|r' \times r''|}{|r'|^3} |r'|^2 = \frac{|r' \times r''|}{|r'|}$.

Fact

We can decompose acceleration into its tangential and normal components as

$$a = \frac{r' \cdot r''}{|r'|} T + \frac{|r' \times r''|}{|r'|} N.$$

Before discussing Kepler's laws, we should review ellipses.

Definition

An ellipse is a set of points the sum of whose distances from two fixed Foci F_1 and F_2 is constant.

Fact

Suppose that the two foci are placed at $(\pm c, 0)$ *and that the constant sum of distances is* $2a$. *Then the points on the ellipse described above satisfy*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Fact (Alternative Definition)

Let F be a fixed point (focus) and let ℓ be a fixed line (direction) in a plane. Let e be a fixed positive number (eccentricity). The set of all points P satisfying $\frac{|PF|}{|P\ell|} = e$ is an ellipse if $e < 1$ (a parabola if $e = 1$ and a hyperbola if $e > 1$).

Fact

The polar equation for the curve described above is

$$r = \frac{ed}{1 + e\cos(\theta)},$$

where $d = |F\ell|$.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Newton's Laws of Motion and Gravitation

2nd Law of Motion $F = ma$.

Gravitation $F = \frac{-GMm}{|r|^3} r = \frac{-GMm}{|r|^2} u$.

From Newton's Laws, we have

$$a = \frac{-GM}{|r|^3} r.$$

Thus r and a are parallel which implies that $r \times a = 0$.
We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}(r \times v) &= r' \times v + r \times v' \\ &= v \times v + r \times a \\ &= 0. \end{aligned}$$

Thus, $r \times v = h$, where h is a constant.
We may assume that $h \neq 0$, that is that r and v are not parallel.
Note that this means that $r(t) \perp h$ for all t.
So, the orbit of the planet lies in a plane with normal vector h.

Let's rewrite h as

$$\begin{aligned} h &= r \times v = r \times r' = |r|u \times (|r|u)' \\ &= |r|u \times (|r|'u + |r|u') = |r||r|'(u \times u) + |r|^2(u \times u') \\ &= |r|^2(u \times u'). \end{aligned}$$

Then,

$$\begin{aligned} a \times h &= \frac{-GM}{|r|^2} u \times (|r|^2 u \times u') = -GMu \times (u \times u') \\ &= -GM[(u \cdot u')u - (u \cdot u)u'] \\ &= GMu'. \end{aligned}$$

Thus, $(v \times h)' = a \times h = GMu'$.
Integrating both sides gives $v \times h = GMu + c$, where c is a constant vector.

Note that $v \times h, u \perp h$. Thus, c is in the plane normal to h.
So we will choose coordinate axes so that k points in the direction of h and so that i points in the direction of c.
Now, letting θ denote the angle between c and r,
$(|r|, \theta)$ are the polar coordinates of the planet in the xy-plane.
We now have,

$$\begin{aligned} r \cdot (v \times h) &= r \cdot (GMu + c) = GMr \cdot u + r \cdot c \\ &= GM|r|u \cdot u + |r||c|\cos(\theta) = GM|r| + |r||c|\cos(\theta) \end{aligned}$$

Solving for $|r|$, we have

$$|r| = \frac{r \cdot (v \times h)}{GM + |c|\cos(\theta)}.$$

Noting that $r \cdot (v \times h) = (r \times v) \cdot h = h \cdot h = |h|^2$, we have

$$|r| = \frac{|h|^2/GM}{1 + |c|/GM\cos(\theta)}.$$

Now, letting $e = \frac{|c|}{GM}$ and $d = \frac{|h|^2}{|c|}$, we see that the polar coordinates $(|r|, \theta)$, must satisfy

$$|r| = \frac{de}{1 + e\cos(\theta)},$$

which is the polar coordinates equation for a conic section. Since, the orbit of a planet is a closed curve, we deduce that $e < 1$ and that the curve is an ellipse.