MTHSC 206 Section 14.2 – Limits and Continuity

Kevin James

Example

Consider the function $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ near $(0,0)$.

$x \backslash y$	-1.000	-0.667	-0.333	0.000	0.333	0.667	1.000
-1.000	0.455	0.687	0.807	0.841	0.807	0.687	0.455
-0.667	0.687	0.873	0.949	0.967	0.949	0.873	0.687
-0.333	0.807	0.949	0.992	0.998	0.992	0.949	0.807
0.000	0.841	0.967	0.998	-	0.998	0.967	0.841
0.333	0.807	0.949	0.992	0.998	0.992	0.949	0.807
0.667	0.687	0.873	0.949	0.967	0.949	0.873	0.687
1.000	0.455	0.687	0.807	0.841	0.807	0.687	0.455

It seems reasonable to say that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Example

Consider the function $g(x, y) = \frac{(x^2 - y^2)^2}{(x^2 + y^2)^2}$.

$x \backslash y$	-1.000	-0.667	-0.333	-0.000	0.333	0.667	1.000
-1.000	0.000	0.148	0.640	1.000	0.640	0.148	0.000
-0.667	0.148	0.000	0.360	1.000	0.360	0.000	0.148
-0.333	0.640	0.360	0.000	1.000	0.000	0.360	0.640
-0.000	1.000	1.000	1.000	-	1.000	1.000	1.000
0.333	0.640	0.360	0.000	1.000	0.000	0.360	0.640
0.667	0.148	0.000	0.360	1.000	0.360	0.000	0.148
1.000	0.000	0.148	0.640	1.000	0.640	0.148	0.000

Here we say that $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exist.
For a better understanding of the problem try approaching $(0, 0)$ along the line $y = mx$.

Definition

Suppose that $f(x, y)$ has domain D and that (a, b) is in the interior of D. Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and if $|(x, y), (a, b)| < \delta$, then $|f(x, y) - L| < \epsilon$.

Note

Suppose that $f(x, y)$ is a function of two variables and that C_1 and C_2 are two curves which intersect at (a, b). If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along C_2 and if $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example

Consider the function $f(x,y) = \frac{xy^2}{x^2+y^4}$. Does $\lim_{(x,y)\to(0,0} f(x,y)$ exit?

Example

Consider the function $f(x,y) = \frac{x^2y}{x^2+y^2}$. Compute $\lim_{(x,y)\to(0,0} f(x,y)$.

Note

1. $\lim_{(x,y)\to(a,b)} x = a$, $\lim_{(x,y)\to(a,b)} y = b$, $\lim_{(x,y)\to(a,b)} c = c$.
2. $\lim_{(x,y)\to(a,b)}(f(x,y)+g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$.
3. $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$.
4. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$.

Definition

We say that $f(x, y)$ is <u>continuous at (a, b)</u> if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

We say that f is <u>continuous on D</u> is f is continuous at each point of D.

Note

Using properties of limits, one can prove that if f and g are continuous on their domains then so are $f + g$, $f - g$, fg and f/g. Note that any zeros of g will not be in the domain of f/g.

Fact

Polynomials and rational functions (ratios of polynomials) are continuous on their domain.

Example

1. Consider the function $g(x,y) = \frac{(x^2-y^2)^2}{(x^2+y^2)^2}$. Where is g continuous?
2. What about the function
$$h(x,y) = \begin{cases} \frac{(x^2-y^2)^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0)? \end{cases}$$
3. Where is the function
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
continuous?

Fact

Suppose that $f(x, y)$ and $g(t)$ are continuous functions with g defined on the range of f. Then $g \circ f$ is continuous on the domain of f.

Example

1. Where is the function $h(x, y) = |(x, y)|$ continous?
2. Where is the function $\sin(y/x)$ continuous?

Definition

If f is a real valued function defined on $D \subseteq \mathbb{R}^n$, then we say that $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if for all $\epsilon > 0$, there is $\delta > 0$ such that whenever $|\vec{x} - \vec{a}| < \delta$, $|f(\vec{x}) - L| < \epsilon$, where $|\vec{x} - \vec{a}| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$.

Definition

Suppose that f is a real valued function defined on $D \subseteq \mathbb{R}^n$. Then we say that f is continuous at $\vec{a} \in \mathbb{R}^n$ if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a}).$$