# MTHSC 206 Section 14.2 – Limits and Continuity

Kevin James

## Example

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|---|---|---|---|---|---|---|---|
| -1.000 | 0.455 | 0.687 | 0.807 | 0.841 | 0.807 | 0.687 | 0.455 |
| -0.667 | 0.687 | 0.873 | 0.949 | 0.967 | 0.949 | 0.873 | 0.687 |
| -0.333 | 0.807 | 0.949 | 0.992 | 0.998 | 0.992 | 0.949 | 0.807 |
| 0.000 | 0.841 | 0.967 | 0.998 | - | 0.998 | 0.967 | 0.841 |
| 0.333 | 0.807 | 0.949 | 0.992 | 0.998 | 0.992 | 0.949 | 0.807 |
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It seems reasonable to say that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

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| -1.000 | 0.000 | 0.148 | 0.640 | 1.000 | 0.640 | 0.148 | 0.000 |
| -0.667 | 0.148 | 0.000 | 0.360 | 1.000 | 0.360 | 0.000 | 0.148 |
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Here we say that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.
For a better understanding of the problem try approaching $(0,0)$ along the line $y = mx$.

## Definition

Suppose that $f(x, y)$ has domain $D$ and that $(a, b)$ is in the interior of $D$. Then we say that the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ is $L$ and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and if $|(x, y), (a, b)| < \delta$, then $|f(x, y) - L| < \epsilon$.

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## Note

Suppose that $f(x, y)$ is a function of two variables and that $C_1$ and $C_2$ are two curves which intersect at $(a, b)$. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along $C_1$ and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along $C_2$ and if $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

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Consider the function $f(x, y) = \frac{xy^2}{x^2+y^4}$. Does $\lim_{(x,y)\to(0,0} f(x, y)$ exit?

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1. $\lim_{(x,y)\to(a,b)} x = a$, $\lim_{(x,y)\to(a,b)} y = b$, $\lim_{(x,y)\to(a,b)} c = c$.

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2. $\lim_{(x,y)\to(a,b)}(f(x,y) + g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$.

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3. $\lim_{(x,y)\to(a,b)} f(x, y)g(x, y) = \lim_{(x,y)\to(a,b)} f(x, y) \cdot \lim_{(x,y)\to(a,b)} g(x, y)$.

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4. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$.

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## Note

Using properties of limits, one can prove that if $f$ and $g$ are continuous on their domains then so are $f + g$, $f - g$, $fg$ and $f/g$. Note that any zeros of $g$ will not be in the domain of $f/g$.

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## Fact

*Polynomials and rational functions (ratios of polynomials) are continuous on their domain.*

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$$h(x,y) = \begin{cases} \frac{(x^2-y^2)^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0)? \end{cases}$$

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## Example

1. Where is the function $h(x, y) = |(x, y)|$ continous?
2. Where is the function $\sin(y/x)$ continuous?

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If $f$ is a real valued function defined on $D \subseteq \mathbb{R}^n$, then we say that $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if for all $\epsilon > 0$, there is $\delta > 0$ such that whenever $|\vec{x} - \vec{a}| < \delta$, $|f(\vec{x}) - L| < \epsilon$, where
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### Definition

Suppose that $f$ is a real valued function defined on $D \subseteq \mathbb{R}^n$. Then we say that $f$ is continuous at $\vec{a} \in \mathbb{R}^n$ if

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$