# MTHSC 206 Section 14.3 – Partial Derivatives

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### Definition

Suppose that $f(x, y)$ is a function of two variables with domain $D$. Suppose that $(a, b) \in D$. Define $g(x) = f(x, y)$ and $h(y) = f(x, y)$. Then we define the partial derivatives of $f$ with respect to the variables $x$ and $y$ as follows.

$$f_x(x, y) = g'(x), \qquad f_y(x, y) = h'(y).$$

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## Fact

$$\begin{aligned} f_x(a, b) &= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \quad \textit{and} \\ f_y(a, b) &= \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}. \end{aligned}$$

## Notation

$$\begin{aligned} f_x(x,y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f, \\ f_y(x,y) &= f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f. \end{aligned}$$

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## Example

Compute the partial derivatives with respect to $x$ and $y$ for the following functions.

1. $f(x,y) = x^4 + y^3 + 3xy$.
2. $g(x,y) = \sin(xy)$.

## Note

Suppose that $f(x, y)$ is a function with domain $D$ and that $(a, b) \in D$. Consider the curves $C_x : z = f(x, b)$ and $C_y : z = f(a, y)$. The slope of the line in the plane $y = b$ which is tangent to $C_x$ at $(a, b, (f(a, b))$ is $f_x(a, b)$.

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The slope of the line in the plane $x = a$ which is tangent to $C_y$ at $(a, b, f(a, b))$ is $f_y(a, b)$.
The equations of the tangent lines are

$$\begin{cases} z - c = f_x(a, b)(x - a) \\ y = b. \end{cases} \qquad \begin{cases} z - c = f_y(a, b)(y - b) \\ x = a. \end{cases}$$

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They can be parametrized by the vector functions

$$\begin{aligned} r_x(t) &= (a, b, c) + t(1, 0, f_x(a, b)) \quad \text{and} \\ r_y(t) &= (a, b, c) + t(0, 1, f_y(a, b)). \end{aligned}$$

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2. Find the equations of the tangent lines to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$.

## Note

We can of course consider the partial derivatives of functions of many variables. Suppose that $f(\vec{x})$ is a function of the $n$ variables $x_1, \ldots, x_n$. Then we define

$$f_{x_i}(\vec{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}.$$

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## Example

Compute the partial derivatives of $f(x, y, z) = x \sin(y) e^z$.

## Note

We can of course consider higher derivatives. The notation for the second order derivatives is as follows.

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\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2},\\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},\\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y},\\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.
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The notation for third and higher order derivatives is similar.

**Example**

Let $f(x, y) = x^2 - xy + y^2$. Compute the 2nd order partial derivatives of $f$.

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### Theorem (Clairaut)

*Suppose that $f$ is defined at $(a, b)$. If there is a disk $D$ containing $(a, b)$ on which $f_{xy}$ and $f_{yx}$ are continuous, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$