MTHSC 206 Section 14.4 – Tangent Planes and Linear Approximations

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Fact

Suppose that f has continuous partial derivatives at (x_0, y_0). Then the tangent plane of f at (x_0, y_0) (i.e. the plane determined by the tangent lines to $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$. has the equation

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

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Example

Find the equation of the plane tangent to $z = 3x^2 + 5y^2$ at the point $(1, 1, 8)$.

Definition

Suppose that f has continuous partial derivatives. We define the <u>linearization of f at (x_0, y_0)</u> as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

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Example

Approximate the value of $3x^2 + 5y^2$ at $(1.01, 1.01)$.

Definition

The function $z = f(x, y)$ is differentiable at (a, b) if the quantity $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

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Theorem

If f_x and f_y are defined and continuous near (a, b), then f is differentiable at (a, b).

Example

Show that $f(x, y) = y\sin(xy)$ is differentiable at $(\frac{1}{2}, \pi)$. Find its linearization at $(\frac{1}{2}, \pi)$ and use it to approximate $f(.55, 3)$.

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Definition

Given a function $f(x,y)$, we define differentials dx and dy to be independent variables. We define the total differential dz as

$$\mathrm{dz} = f_x(x,y)\mathrm{dx} + f_y(x,y)\mathrm{dy}.$$

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In the above expression, the dependent variable dz depends on the 4 independent variables $x, y,$ dx and dy.

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Note

If f has continuous partials, then $f(x + \mathrm{dx}, y + \mathrm{dy}) \approx f(x, y) + \mathrm{dz}$.

Example

Suppose that we are to construct a cylindrical water tank with radius 2m and height 1m. Suppose also that we can ensure that the radius and height of the tank are correct to within 1mm. What is the maximum error in the volume of the tank.

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Fact

If $f(\vec{x})$ is a real valued function of n variables, then we consider the differentials dx_i as independent variables and define the total differential to be

$$dz = \sum_{i=1}^{n} f_{x_i}(\vec{x})dx_i.$$

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$$dz = \sum_{i=1}^{n} f_{x_i}(\vec{x})dx_i.$$

If f is differentiable, then it will be true that

$$f(\vec{x} + \vec{dx}) \approx f(\vec{x}) + dz.$$