MTHSC 206 Section 14.5 – The Chain Rule

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Theorem (Chain Rule - Case 1)

Suppose that $z = f(x, y)$ is a differentiable function and that $x(t)$ and $y(t)$ are both differentiable functions as well. Then,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

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Example

Suppose that $z = xy^2 + 5x^3y$ where $x(t) = e^t$ and $y(t) = \sin(t)$. Find $\frac{\mathrm{dz}}{\mathrm{dt}}$ when $t = 0$.

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The pressure (in kilopascals kPa), volume (in liters L) and temperature (in kelvins K) of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at 0.2 L/s.

Theorem (Chain Rule - Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x(s, t)$ and $y(s, t)$ are also differentiable functions. Thesn

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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Example

Suppose that $z = \cos(x)\sin(y)$ and $x(s, t) = st$ and $y(s, t) = s^2 t$. Compute the partial derivatives of z with respect to s and t.

Theorem (Chain Rule - General Version)

Suppose that u is a differentiable function in the variables $x_1, x_2, \ldots, x_n$ and each x_i is a differentiable function of the variables $t_1, t_2, \ldots, t_m$. Then,

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial t_i}.$$

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Example

Suppose that $u = x^3 y + y^3 z + z^3 x$ where $x = rs\sin(t)$, $y = rs\cos(t)$ and $z = rse^t$. Find $\frac{\partial u}{\partial s}$.

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\frac{\partial F}{\partial x}\frac{\mathrm{dx}}{\mathrm{dx}} + \frac{\partial F}{\partial y}\frac{\mathrm{dy}}{\mathrm{dx}} &= 0 \\
\Rightarrow \frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\frac{-\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}
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EXAMPLE

Find the slope of the line tangent to the unit circle at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

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\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0\\
\Rightarrow \frac{\partial z}{\partial x} &= \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\\
&= \frac{-F_x}{F_z}
\end{aligned}$$

Implicit Differentiation

Suppose that x, y and z satisfy the equation $F(x, y, z) = 0$ where F is differentiable then under the assumption that z is implicitly defined as a differentiable function of x and$y$, we obtain the formulas

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\\ \frac{\partial z}{\partial y} &= \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\end{aligned}$$

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$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example

Recall that the equation of the unit sphere is given by $x^2 + y^2 + z^2 = 1$. Use implicit differentiation to find the equation of the tangent plane at the point $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$