

# MTHSC 206 SECTION 14.7 – MAXIMUM AND MINIMUM VALUES

Kevin James

## DEFINITION

- We say that a function  $f(x, y)$  has a local maximum at  $(a, b)$  if there is a disk  $D$  centered at  $(a, b)$  for which  $f(x, y) \leq f(a, b)$  for all  $(x, y)$  in  $D$ . In this case,  $f(a, b)$  is called a local maximum value of  $f$ .
- We say that a function  $f(x, y)$  has a local minimum at  $(a, b)$  if there is a disk  $D$  centered at  $(a, b)$  for which  $f(x, y) \geq f(a, b)$  for all  $(x, y)$  in  $D$ . In this case,  $f(a, b)$  is called a local minimum value of  $f$ .
- Suppose that  $f(x, y)$  has domain  $\mathcal{D}$ . We say that  $f(x, y)$  has an absolute maximum at  $(a, b)$  if  $f(x, y) \leq f(a, b)$  for all  $(x, y)$  in  $\mathcal{D}$ . In this case,  $f(a, b)$  is called an absolute maximum value of  $f$ .
- We say that  $f(x, y)$  has an absolute minimum at  $(a, b)$  if  $f(x, y) \geq f(a, b)$  for all  $(x, y)$  in  $\mathcal{D}$ . In this case,  $f(a, b)$  is called an absolute minimum value of  $f$ .

## THEOREM

Suppose that  $f(x, y)$  has a local extreme at  $(a, b)$  and that  $f_x$  and  $f_y$  exist at  $(a, b)$ . Then  $f_x(a, b) = f_y(a, b) = 0$ .

## DEFINITION

Suppose that  $f(x, y)$  has domain  $\mathcal{D}$  and that  $(a, b) \in \mathcal{D}$ . We say that  $(a, b)$  is a critical point if  $f_x(a, b) = f_y(a, b) = 0$  or if one of  $f_x$  or  $f_y$  does not exist at  $(a, b)$ .

## NOTE

From our theorem above, we see that local extrema occur at critical points. However, it is not necessarily true that all critical points are locations of local extrema.

### EXAMPLE

Consider  $f(x, y) = x^2 + y^2 - 2x + 6y + 10$ . Identify the critical points and discuss the possibility of local extrema.

### EXAMPLE (BEWARE SADDLE POINTS)

Consider  $f(x, y) = y^2 - x^2$ . Identify the critical points and discuss the possibility of local extrema.

## THEOREM (2ND DERIVATIVE TEST)

Suppose that the second order partials of  $f(x, y)$  exist and are continuous on a disk centered at  $(a, b)$ . Further suppose that  $f_x(a, b) = f_y(a, b) = 0$ . Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

- 1 If  $D > 0$  and  $f_{xx}(a, b) > 0$  then  $(a, b)$  is a local minimum.
- 2 If  $D > 0$  and  $f_{xx}(a, b) < 0$  then  $(a, b)$  is a local maximum.
- 3 If  $D < 0$  then  $(a, b)$  is not a local extreme.

## NOTE

- 1 In case 3,  $(a, b)$  is called a saddle point and the graph of  $f(x, y)$  crosses its tangent plane.
- 2 If  $D = 0$  then we have no information,  $(a, b)$  could be a local max or local min or a saddle point.
- 3 Note that  $D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$ .

## EXAMPLE

Find the maximum and minimum values of the function  
 $f(x, y) = x^4 + y^4 - 4xy + 1$ .

## EXAMPLE

Suppose that we wish to construct a box with no lid using  $10m^2$  of cardboard in such a way as to maximize the volume of the box. What dimensions should we choose?

## DEFINITION

- A boundary point of a set  $D$  in  $\mathbb{R}^2$  is a point  $(a, b)$  such that every disk centered at  $(a, b)$  contains points in  $D$  and points not in  $D$ .
- A closed set in  $\mathbb{R}^2$  is a set that contains all of its boundary points.
- A bounded set in  $\mathbb{R}^2$  is a set which is completely contained in some disk.
- A set in  $\mathbb{R}^2$  which is both closed and bounded is called compact.

## THEOREM (EXTREME VALUES)

*If  $f$  is continuous on a compact set  $D$  in  $\mathbb{R}^2$ , then  $f$  attains absolute maximum and absolute minimum values on  $D$ .*

## FINDING ABSOLUTE EXTREMA ON A COMPACT SET

We use the following procedure to find the absolute extreme values of a continuous function  $f$  on a compact set  $D$ .

- 1 Find the values of  $f$  at the critical points of  $f$  in  $D$ .
- 2 Find the extreme values of  $f$  on the boundary of  $D$ .
- 3 The largest of the values from the previous steps is the absolute maximum value of  $f$  on  $D$ . The smallest of the values from the previous steps is the absolute minimum value of  $f$  on  $D$ .

### EXAMPLE

Find the extreme values of the function  $f(x, y) = x^2 - 2xy + 2y$  on the rectangle  $D = \{(x, y) \mid 0 \leq x \leq 4, -1 \leq y \leq 3\}$ .