MTHSC 206 Section 14.7 – Maximum and Minimum Values

Kevin James

Definition

- We say that a function $f(x, y)$ has a local maximum at (a, b) if there is a disk D centered at (a, b) for which $f(x, y) \leq f(a, b)$ for all (x, y) in D. In this case, $f(a, b)$ is called a local maximum value of f.

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- We say that a function $f(x, y)$ has a local minimum at (a, b) if there is a disk D centered at (a, b) for which $f(x, y) \geq f(a, b)$ for all (x, y) in D. In this case, $f(a, b)$ is called a local minimum value of f.

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- Suppose that $f(x, y)$ has domain $\mathcal{D}$. We say that $f(x, y)$ has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in $\mathcal{D}$. In this case, $f(a, b)$ is called an absolute maximum value of f.

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Suppose that $f(x, y)$ has a local extreme at (a, b) and that f_x and f_y exist at (a, b). Then $f_x(a, b) = f_y(a, b) = 0$.

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Note

From our theorem above, we see that local extrema occur at critical points. However, it is not necessarily true that all critical points are locations of local extrema.

Example

Consider $f(x, y) = x^2 + y^2 - 2x + 6y + 10$. Identify the critical points and discuss the possibility of local extrema.

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Example (Beware Saddle Points)

Consider $f(x, y) = y^2 - x^2$. Identify the critical points and discuss the possibility of local extrema.

Theorem (2nd Derivative Test)

Suppose that the second order partials of $f(x, y)$ exist and are continuous on a disk centered at (a, b). Further suppose that $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.*
3. *If $D < 0$ then (a, b) is not a local extreme.*

Note

1. In case 3, (a, b) is called a saddle point and the graph of $f(x, y)$ crosses its tangent plane.
2. If $D = 0$ then we have no information, (a, b) could be a local max or local min or a saddle point.
3. Note that $D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$.

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Find the maximum and minimum values of the function $f(x, y) = x^4 + y^4 - 4xy + 1$.

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Suppose that we wish to construct a box with no lid using $10m^2$ of cardboard in such a way as to maximize the volume of the box. What dimensions should we choose?

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Theorem (Extreme Values)

If f is continuous on a compact set D in $\mathbb{R}^2$, then f attains absolute maximum and absolute minimum values on D.

Finding absolute Extrema on a Compact Set

We use the following procedure to find the absolute extreme values of a continuous function f on a compact set D.

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from the previous steps is the absolute maximum value of f on D. The smallest of the values from the previous steps is the absolute minimum value of f on D.

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Example

Find the extreme values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 4, -1 \leq y \leq 3\}$.