MTHSC 206 Section 14.8 – Lagrange Multipliers

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Goal

We would like to maximize a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$.

Idea

We want to pick the largest value of c for which the level surfaces $f(x, y, z) = c$ and $g(x, y, z) = k$ intersect.
Motivated by the two variable case, we expect this to happen when the surfaces are tangent,
and he extreme value occurs at the point of tangency, say (x_0, y_0, z_0).
Thus, the normal lines of our surfaces at (x_0, y_0, z_0) should be equal.
This implies that $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ should be parallel.
Thus there should be a constant $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

Definition

We say that λ is a Lagrange multiplier if there are x_0, y_0 and z_0 such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

Method of Lagrange Multipliers

To find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ under the assumption that extremes exist and that $\nabla g \neq 0$ on the surface $g(x, y, z) = k$, we follow the following steps.

1. Find all values of x, y, z and λ such that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k. \end{aligned}$$

2. Evaluate f at these points. The smallest value is the minimum and the largest is the maximum.

Note

It is not always necessary to find λ.

Example

Suppose that we wish to construct a box with no lid using $10m^2$ of cardboard in such a way as to maximize the volume of the box. What dimensions should we choose?

Example

Find the points on the unit sphere which are closest to and farthest from the point $(1, 1, 2)$.

Idea

Suppose that we want to find the extremes of $f(x, y, z)$ subject to the two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$.
That is, we want to find the extremes of f along the curve C of intersection of the two surfaces $g = k$ and $h = c$.
Note that ∇g and ∇h are orthogonal to C.
We will suppose $\nabla g, \nabla h \neq 0$ and not parallel on C.
Suppose f attains an extreme at a point P, then $\nabla g(P)$ and $\nabla h(P)$ determine a plane which contains all vectors normal to C at P

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Since f attains an extreme value on C at $P = (x_0, y_0, z_0)$ then ∇f will be orthogonal to C at P.
Thus $\nabla f(P)$ is in the plane determined by $\nabla g(P)$ and $\nabla h(P)$.
Thus there exist Lagrange multipliers λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

Example

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.