MTHSC 206 Section 14.8 – Lagrange Multipliers

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Thus there should be a constant $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

Definition

We say that λ is a Lagrange multiplier if there are x_0, y_0 and z_0 such that

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Method of Lagrange Multipliers

To find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ under the assumption that extremes exist and that $\nabla g \neq 0$ on the surface $g(x, y, z) = k$, we follow the following steps.

1. Find all values of x, y, z and λ such that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k. \end{aligned}$$

2. Evaluate f at these points. The smallest value is the minimum and the largest is the maximum.

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Find the points on the unit sphere which are closest to and farthest from the point $(1, 1, 2)$.

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Thus there exist Lagrange multipliers λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

Example

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.